

Foldy-Wouthuysen transformation for a Dirac-Pauli dyon and the Thomas-Bargmann-Michel-Telegdi equation

Tsung-Wei Chen^{1,*} and Dah-Wei Chiou^{2,1,†}

¹*Department of Physics and Center for Theoretical Sciences,
National Taiwan University, Taipei 106, Taiwan*

²*Department of Physics, Beijing Normal University, Beijing 100875, China*

The classical dynamics for a charged point particle with intrinsic spin is governed by a relativistic Hamiltonian for the orbital motion and by the Thomas-Bargmann-Michel-Telegdi equation for the precession of the spin. It is natural to ask whether the classical Hamiltonian (with both the orbital and spin parts) is consistent with that in the relativistic quantum theory for a spin-1/2 charged particle, which is described by the Dirac equation. In the low-energy limit, up to terms of the 7th order in $1/E_g$ ($E_g = 2mc^2$ and m is the particle mass), we investigate the Foldy-Wouthuysen (FW) transformation of the Dirac Hamiltonian in the presence of homogeneous and static electromagnetic fields and show that it is indeed in agreement with the classical Hamiltonian with the gyromagnetic ratio being equal to 2. Through electromagnetic duality, this result can be generalized for a spin-1/2 dyon, which has both electric and magnetic charges and thus possesses both intrinsic electric and magnetic dipole moments. Furthermore, the relativistic quantum theory for a spin-1/2 dyon with arbitrary values of the gyromagnetic and gyroelectric ratios can be described by the Dirac-Pauli equation, which is the Dirac equation with augmentation for the anomalous electric and anomalous magnetic dipole moments. The FW transformation of the Dirac-Pauli Hamiltonian is shown, up to the 7th order again, to be also in accord with the classical Hamiltonian.

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I. INTRODUCTION

The relativistic quantum theory for a spin-1/2 point particle is described by the Dirac equation [1]. The wavefunction used for the Dirac equation is the Dirac bispinor, which is composed of two Weyl spinors corresponding to the particle and antiparticle parts. Rigorously, the Dirac equation is self-consistent only in the context of quantum field theory, in which the particle-antiparticle pairs can be created. In the low-energy limit, if the relevant energy (the particle's energy interacting with electromagnetic fields) is much smaller than the Dirac energy gap $E_g = 2mc^2$ (m is the particle mass), the probability of creation of particle-antiparticle pairs is negligible and the Dirac equation is adequate to describe the relativistic quantum dynamics of the spin-1/2 particle without taking into account the field-theory interaction to the antiparticle.

The Foldy-Wouthuysen (FW) transformation is one of the methods developed to investigate the low-energy limit of the Dirac equation [2].¹ In the FW method, $1/E_g$ is

treated as the small parameter; the Dirac Hamiltonian in the Dirac bispinor representation is block diagonalized up to a certain order of $1/E_g$ and the remaining off-diagonal matrices, which correspond to the particle-antiparticle interactions, are brought into the next order of $1/E_g$ and thus neglected. This is achieved by a series of successive unitary transformations performed on the Dirac Hamiltonian. Furthermore, a series of successive transformation in FW method can be reduced into one single transformation by the use of the Löwding partitioning method [3]. For a charged spin-1/2 particle subject to a non-explicitly time-dependent field, an exact FW transformation has been found by Eriksen [4], and the validity of the transformation is studied in [5].

Alternatively, the Dirac Hamiltonian can also be expanded in powers of Plank constant \hbar [6]. In this approach, the small parameter is not the particle's energy (divided by E_g), but its wave length. A diagonalization procedure based on the expansion in powers of \hbar has been constructed in [7, 8]. In this procedure, the Berry phase correction can also be taken into account. Furthermore, the semiclassical \hbar -expansion enables us to describe the quantum corrections on the classical expression in strong fields [9].

On the other hand, the classical (non-quantum) dynamics for a relativistic point particle endowed with charge and intrinsic spin in static and homogeneous electromagnetic fields is well understood. The orbital motion is govern by the relativistic Hamiltonian and the precession of the spin by the Thomas-Bargmann-Michel-Telegdi (TBMT) equation [10]. The relativistic Hamiltonian for the orbital motion plus the Hamiltonian obtained from the TBMT equation (called TBMT Hamil-

*Electronic address: twchen@phys.ntu.edu.tw

†Electronic address: chiou@gravity.psu.edu

¹ It is often said that FW method gives the nonrelativistic limit of the Dirac equation. The phrase “nonrelativistic” is somewhat misleading as it usually refers to “low-speed” limit. As we will show in this paper, the FW transformation (if performed to orders high enough) actually agrees with the relativistic classical dynamics even when the speed of the particle is large. The appropriate description is to say that the FW transformation yields “low-energy” limit.

tonian) is expected to provide a low-energy description of the relativistic quantum theory. The conjecture that the low-energy limit of the Dirac Hamiltonian reduces to the classical orbital Hamiltonian plus the TBMT Hamiltonian has been suggested but remains to be affirmed.

In order to investigate the consistency between the low-energy limit of the Dirac equation and the classical dynamics, we perform a series of FW transformations and expand the Dirac Hamiltonian up to terms of the 7th order in $1/E_g$. The electromagnetic fields are assumed to be static and homogeneous. Taking care of the relation between the kinematic momentum used in the Dirac Hamiltonian and the boost velocity used in the TBMT Hamiltonian, we show that the FW transformation of the Dirac Hamiltonian is in agreement with the classical orbital Hamiltonian plus the TBMT Hamiltonian for the case of the gyromagnetic ratio equal to 2. Through electromagnetic duality, this result can be generalized for a spin-1/2 dyon [11], which has both electric and magnetic charges and thus possesses both intrinsic electric and magnetic dipole moments (with both gyromagnetic and gyroelectric ratios equal to 2).

To affirm the consistency to a broader extent, we need to show that the relativistic quantum theory of a spin-1/2 dyon with arbitrary values of the gyromagnetic and gyroelectric ratios also reduces to the classical counterparts as a low-energy limit. The relativistic quantum theory of a spin-1/2 dyon with the inclusion of anomalous magnetic dipole moment (AMM) and anomalous electric dipole moment (AEM) can be described by the Dirac-Pauli equation [9, 12], which is the Dirac equation with augmentation for AMM and AEM. The FW transformation is performed on the Dirac-Pauli Hamiltonian, again up to the 7th order in $1/E_g$, and the result confirms that it remains in agreement with the classical orbital Hamiltonian plus the TBMT Hamiltonian for arbitrary values of the gyromagnetic and gyroelectric ratios.

This paper is organized as follows. In Sec. II, we investigate the tensorial structure of the orbital and intrinsic dipole moments. In Sec. III, we briefly review the classical orbital Hamiltonian and the TBMT equation. In Sec. IV, we perform the FW transformation on the Dirac Hamiltonian for a spin-1/2 dyon and show that it agrees with the TBMT equation for the case of the gyromagnetic and gyroelectric ratios equal to 2. Later in Sec. V, we perform the FW transformation on the Dirac-Pauli Hamiltonian and show that it again agrees with the TBMT equation even with the inclusion of AMM and AEM. Finally, the conclusions are summarized and discussed in Sec. VI. Some calculational details are supplemented in Appendices A and B.

II. ORBITAL AND INTRINSIC DIPOLE MOMENTS

For a general Lorentz transformation from the primed (boosted) frame to the unprimed (laboratory) frame, the

transformation of a 4-vector $k^\mu = (k^0, \mathbf{k})$ is given by [13]:

$$\begin{aligned} k^0 &= \gamma(k'^0 + \boldsymbol{\beta} \cdot \mathbf{k}'), \\ \mathbf{k} &= \mathbf{k}' + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{k}')\boldsymbol{\beta} + \gamma k'^0 \boldsymbol{\beta}, \end{aligned} \quad (1)$$

where $\mathbf{v} = c\boldsymbol{\beta}$ is the boost velocity of the primed frame relative to the unprimed frame, γ is the Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$ and $\beta = |\boldsymbol{\beta}|$.

In the primed system, let us consider the case that charge and current densities satisfy the conditions:

$$\int_{V'} d^3x' \rho' = 0, \quad \int_{V'} d^3x' \mathbf{J}' = 0. \quad (2)$$

The vanishing of the total charge means that the system is *neutral*, and the vanishing of the total current is a consequence of the *static* condition: $\partial \rho' / \partial t' = -\nabla' \cdot \mathbf{J}' = 0$.² Because the charge density and current density form a 4-vector $J^\mu = (c\rho, \mathbf{J})$, it can be shown that the same conditions also hold in the unprimed system:

$$\int_V d^3x \rho = 0, \quad \int_V d^3x \mathbf{J} = 0. \quad (3)$$

In the unprimed frame, the magnetic dipole moments \mathbf{m} is defined as

$$\mathbf{m} = \int_V d^3x \boldsymbol{\mu}_m, \quad (4)$$

where

$$\boldsymbol{\mu}_m = \frac{1}{2c}(\mathbf{x} \times \mathbf{J}) \quad (5)$$

is the magnetic dipole moment density, and the electric dipole moment \mathbf{p} is defined as

$$\mathbf{p} = 2 \int_V d^3x \boldsymbol{\mu}_p^c, \quad (6)$$

where

$$\boldsymbol{\mu}_p^c = \frac{1}{2} \mathbf{x} \rho \quad (7)$$

is the *canonical* electric dipole moment density (the extra factor of 2 is introduced for later convenience). In the primed system, the definitions of both dipole moments are the same as those in the unprimed system. Using Eq. (1), Eq. (5) can be written as

$$\begin{aligned} \boldsymbol{\mu}_m &= \boldsymbol{\mu}_m' + \frac{\gamma - 1}{\beta^2} \boldsymbol{\beta} \times (\boldsymbol{\mu}_m' \times \boldsymbol{\beta}) \\ &\quad + \frac{1}{2} \gamma (\mathbf{x}' c \rho' - x'^0 \mathbf{J}') \times \boldsymbol{\beta}, \end{aligned} \quad (8)$$

² Since the static condition gives $\nabla' \cdot \mathbf{J}' = 0$, it can be shown $J'_i = \nabla' \cdot (x'_i \mathbf{J}')$. Consequently, $\int_{V'} d^3x' J'_i = \int_{V'} d^3x' \nabla' \cdot (x'_i \mathbf{J}') = \int_{\partial V'} d\mathbf{a}' \cdot (x'_i \mathbf{J}') = 0$ if the current \mathbf{J}' is localized.

where $\boldsymbol{\mu}'_m = \mathbf{x}' \times \mathbf{J}'/2$ is the magnetic dipole density in the primed system. It is interesting to note that if we integrate the term $\mathbf{x}'c\rho' - x'^0\mathbf{J}'$ in the primed system, we obtain

$$\int_{V'} d^3x'(\mathbf{x}'c\rho' - x'^0\mathbf{J}') = \int_{V'} d^3x'\mathbf{x}'c\rho' = c\mathbf{p}', \quad (9)$$

where the neutral condition [Eq. (2)] has been used. This suggests that we can define the *tensorial* electric dipole moment as

$$\boldsymbol{\mu}_p = \frac{1}{2c}(\mathbf{x}J^0 - x^0\mathbf{J}) \quad (10)$$

so that the dipole moment can be defined as a second rank antisymmetric tensor:

$$M^{\mu\nu} = \frac{1}{2c}(x^\mu J^\nu - x^\nu J^\mu). \quad (11)$$

The canonical and tensorial dipole moments densities yield the same (integrated) dipole moments, because the neutral condition ensures that the integration of the second term $x^0\mathbf{J}$ vanishes in the unprimed system. The components of the second rank tensor $M^{\mu\nu}$ are

$$\begin{aligned} M^{0i} &= \frac{1}{2c}(x^0 J^i - x^i J^0) = -\mu_p^i, \\ M^{ij} &= \frac{1}{2c}(x^i J^j - x^j J^i) = \epsilon_{ijk}\mu_m^k. \end{aligned} \quad (12)$$

Consequently, the Lorentz transformation between $(\boldsymbol{\mu}_p, \boldsymbol{\mu}_m)$ and $(\boldsymbol{\mu}'_p, \boldsymbol{\mu}'_m)$ is of the form

$$\begin{aligned} \boldsymbol{\mu}_p &= \gamma(\boldsymbol{\mu}'_p + \boldsymbol{\beta} \times \boldsymbol{\mu}'_m) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{\mu}'_p), \\ \boldsymbol{\mu}_m &= \gamma(\boldsymbol{\mu}'_m - \boldsymbol{\beta} \times \boldsymbol{\mu}'_p) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{\mu}'_m). \end{aligned} \quad (13)$$

The transformation [Eq. (13)] is exactly the same as that for the electric and magnetic fields if we take the replacement rules: $\mathbf{E} \leftrightarrow \boldsymbol{\mu}_p$, $\mathbf{B} \leftrightarrow -\boldsymbol{\mu}_m$.

The corresponding transformation for the (integrated) dipole moments is given by

$$\begin{aligned} \frac{\mathbf{p}}{2} &= \gamma^2 \left[\frac{\mathbf{p}'}{2} + \boldsymbol{\beta} \times \mathbf{m}' - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \frac{\mathbf{p}'}{2}) \right], \\ \mathbf{m} &= \gamma^2 \left[\mathbf{m}' - \boldsymbol{\beta} \times \frac{\mathbf{p}'}{2} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{m}') \right], \end{aligned} \quad (14)$$

where $d^3x = \gamma d^3x'$ is used. Since the extra factor γ arises in the right hand side of Eq. (14) due to the spatial integral, $\mathbf{p}/2$ and \mathbf{m} do not transform covariantly and thus do not form a second rank tensor unlike $\boldsymbol{\mu}_p$ and $\boldsymbol{\mu}_m$.

In the case with only proper electric dipole moment and no proper magnetic dipole moment (i.e. $\mathbf{p}' \neq 0$ and $\mathbf{m}' = 0$), when boosted, the electric dipole will result in a magnetic dipole moment $\mathbf{m} = -\gamma^2(\boldsymbol{\beta} \times \mathbf{p}'/2)$ in the unprimed system. This can be understood as follows: If we

think \mathbf{p}' as two endpoints separated by a short distance and charged with $+q$ and $-q$, in the unprimed system, the positive and negative charges acquire a velocity and give rise to currents in opposite directions, thus resulting in a magnetic dipole moment. In an inhomogeneous magnetic field, a moving object with only proper electric dipole moment can feel the magnetic force $\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B}$.

On the other hand, in the case with only proper magnetic dipole moment and no proper electric dipole moment (i.e. $\mathbf{m}' \neq 0$ and $\mathbf{p}' = 0$), when boosted, the magnetic dipole will result in an electric dipole moment $\mathbf{p} = 2\gamma^2(\boldsymbol{\beta} \times \mathbf{m}')$ in the unprimed system. This is due to the fact that in the unprimed system the charge density ρ arises through the Lorentz transformation even if the charge density is zero in the primed system ($\rho' = 0$). The charge density in the primed system $c\rho = \gamma\boldsymbol{\beta} \cdot \mathbf{J}'$ is positive (negative) when the current is parallel (anti-parallel) to the boost velocity; therefore, as a magnetic dipole can be thought as a small current loop, the small current loop in the primed system gives rise to opposite charges separated by a short distance in the unprimed system, thus resulting in an electric dipole moment. In an inhomogeneous electric field, a moving object with only proper magnetic dipole moment can feel the electric force $\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$.

The dipole moments considered above are *orbital* in the sense that they are sourced by the orbital distribution of $J^\mu(x)$. On the other hand, a point particle can give rise to an *intrinsic* dipole moment if it is charged and endowed with intrinsic spin. The fact that $\boldsymbol{\mu}_p$ and $\boldsymbol{\mu}_m$ form an antisymmetric tensor $M^{\mu\nu}$ suggests that the intrinsic spin \mathbf{s} can be generalized to a second-rank antisymmetric tensor $S^{\mu\nu}$, which gives the intrinsic dipole moments as

$$M^{\mu\nu} = \frac{ge}{2mc} S^{\mu\nu}, \quad (15)$$

where e is the electric charge of the particle, m the mass and g_e the *gyromagnetic ratio*. The spin has only three independent components; thus $S^{\mu\nu}$ is dual to an axial 4-vector $S^\alpha = (S^0, \mathbf{S})$ via

$$S^{\mu\nu} = \frac{1}{c} \epsilon^{\mu\nu\alpha\beta} U_\alpha S_\beta \quad (16)$$

and conversely

$$S^\alpha = \frac{1}{2c} \epsilon^{\alpha\beta\gamma\delta} U_\beta S_{\gamma\delta}, \quad (17)$$

where U^α is the particle's 4-velocity. The 4-vector S^α reduces to the spin \mathbf{s} in the particle's rest frame; i.e., $S'^\alpha = (S'^0, \mathbf{S}') = (0, \mathbf{s})$. The vanishing of the time-component in the particle's rest frame is imposed by the covariant constraint:

$$U_\alpha S^\alpha = 0. \quad (18)$$

In the particle's rest frame, $U'^\alpha = (c, 0, 0, 0)$ and Eq. (15) yields

$$\boldsymbol{\mu}'_m = \frac{ge}{2mc} \mathbf{s}, \quad \boldsymbol{\mu}'_p = 0. \quad (19)$$

Therefore, the intrinsic spin gives only the proper intrinsic magnetic dipole and no proper intrinsic electric dipole. In order to have both proper intrinsic magnetic and electric dipoles, we consider a *dyon* particle [11], which possesses both electric charge e and magnetic charge \tilde{e} , and Eq. (15) is generalized as

$$M^{\mu\nu} = M_e^{\mu\nu} + M_{\tilde{e}}^{\mu\nu} = \frac{g_e e}{2mc} S^{\mu\nu} + \frac{g_{\tilde{e}} \tilde{e}}{2mc} \tilde{S}^{\mu\nu}, \quad (20)$$

where $g_{\tilde{e}}$ is the *gyroelectric ratio* and

$$\tilde{S}^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} S_{\alpha\beta} \quad (21)$$

is the dual of $S^{\mu\nu}$. In the rest frame, Eq. (20) yields both magnetic and electric dipoles:

$$\boldsymbol{\mu}_m^e = \frac{g_e e}{2mc} \mathbf{s}, \quad \boldsymbol{\mu}_p^{\tilde{e}} = -\frac{g_{\tilde{e}} \tilde{e}}{2mc} \mathbf{s}. \quad (22)$$

III. THE THOMAS-BARGMANN-MICHEL-TELEGDI EQUATION

Consider a relativistic point particle endowed with electric charge and intrinsic spin subject to static and homogeneous electromagnetic fields. The orbital motion of the particle is described by

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta \quad (23)$$

and the precession of the spin is govern by the TBMT equation [10]:

$$\frac{dS^\alpha}{d\tau} = \frac{e}{mc} \left[\frac{g_e}{2} F^{\alpha\beta} S_\beta + \frac{1}{c^2} \left(\frac{g_e}{2} - 1 \right) U^\alpha (S_\lambda F^{\lambda\mu} U_\mu) \right]. \quad (24)$$

Equation (23) in the covariant form can be shown to be equivalent to the Hamilton's equations :

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \{\mathbf{x}, H_{\text{orbit}}\}, \\ \frac{d\mathbf{p}}{dt} &= \{\mathbf{p}, H_{\text{orbit}}\} \end{aligned} \quad (25)$$

in the unprimed frame, where \mathbf{p} is the conjugate momentum to \mathbf{x} and the Hamiltonian H_{orbit} governing the orbital motion is given by

$$H_{\text{orbit}}(\mathbf{x}, \mathbf{p}) = \sqrt{(c\mathbf{p} - e\mathbf{A}(\mathbf{x}))^2 + m^2 c^4} + e\phi(\mathbf{x}) \quad (26)$$

with $A^\alpha = (\phi, \mathbf{A})$ being the 4-vector potential for the electromagnetic field $F^{\mu\nu}$. (See Sec. 12.1 in [13] for more details.)

On the other hand, Eq. (24) leads to

$$\frac{d\mathbf{s}}{dt} = \frac{e}{mc} \mathbf{s} \times \mathbf{F}(\mathbf{x}) \quad (27)$$

with

$$\begin{aligned} \mathbf{F} &= \left(\frac{g_e}{2} - 1 + \frac{1}{\gamma} \right) \mathbf{B} - \left(\frac{g_e}{2} - 1 \right) \frac{\gamma}{\gamma + 1} (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta} \\ &\quad - \left(\frac{g_e}{2} - \frac{\gamma}{\gamma + 1} \right) \boldsymbol{\beta} \times \mathbf{E}, \end{aligned} \quad (28)$$

which gives the spin precession with respect to the time of the unprimed frame. (See Sec. 11.11 in [13] for more details.) Because $\{s_i, s_j\} = \epsilon_{ijk} s_k$, Eq. (27) can be recast as the Hamilton's equation:

$$\frac{d\mathbf{s}}{dt} = \{\mathbf{s}, H_{\text{spin}}\} \quad (29)$$

with

$$H_{\text{spin}}(\mathbf{x}, \mathbf{s}) = -\frac{e}{mc} \mathbf{s} \cdot \mathbf{F}(\mathbf{x}) \quad (30)$$

called the TBMT Hamiltonian, which governs the precession of the electric dipole subject to a static and homogeneous field.

In the low-speed limit ($\beta \ll 1$), we have $\gamma \approx 1$ and Eq. (30) gives

$$\begin{aligned} H_{\text{spin}} \approx & -\frac{e}{2mc} \mathbf{s} \cdot \left[g_e \mathbf{B} - \left(\frac{g_e}{2} - 1 \right) (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta} \right. \\ & \left. - (g_e - 1) \boldsymbol{\beta} \times \mathbf{E} \right]. \end{aligned} \quad (31)$$

The first term in Eq. (31) is the interaction energy of the magnetic moment $\boldsymbol{\mu}_m^e$ in the magnetic field, which accounts for the anomalous Zeeman effect. The second term corresponds to the change rate of the longitudinal polarization, which vanishes in the case of $g_e = 2$. The third term is the spin-orbit interaction (the interaction of the boosted electric dipole $\boldsymbol{\mu}_p^e \approx \boldsymbol{\beta} \times \boldsymbol{\mu}_m^e$ coupled to the electric field) plus the correction for the Thomas precession.

By treating \mathbf{x} , \mathbf{p} and \mathbf{s} as independent phase space variables, the total Hamiltonian is given by³

$$H(\mathbf{x}, \mathbf{p}, \mathbf{s}) = H_{\text{orbit}}(\mathbf{x}, \mathbf{p}) + H_{\text{spin}}(\mathbf{x}, \mathbf{s}). \quad (32)$$

If the particle has both electric charge e and magnetic charge \tilde{e} (i.e. the particle is a *dyon*), Eq. (26) and Eq. (30) are modified with the inclusion of the dual counterparts; i.e.

$$\begin{aligned} H_{\text{orbit}}(\mathbf{x}, \mathbf{p}) &= \sqrt{(c\mathbf{p} - e\mathbf{A}(\mathbf{x}) - \tilde{e}\tilde{\mathbf{A}}(\mathbf{x}))^2 + m^2 c^4} \\ &\quad + e\phi(\mathbf{x}) + \tilde{e}\tilde{\phi}(\mathbf{x}) \end{aligned} \quad (33)$$

³ Note that, in order to add H_{orbit} and H_{spin} together, we have to consider $ds/dt \equiv d\mathbf{S}'/dt$ in Eq. (27), instead of $d\mathbf{S}/dt$, $d\mathbf{S}/d\tau$ or $d\mathbf{s}/d\tau$. This is because s_i are degrees of freedom independent of \mathbf{x} and \mathbf{p} , but S_i are not. Furthermore, to be consistent with the orbital motion, the precession is cast with respect to t , instead of the proper time τ of the moving particle.

and

$$H_{\text{spin}}(\mathbf{x}, \mathbf{s}) = -\frac{e}{mc} \mathbf{s} \cdot \mathbf{F}(\mathbf{x}) - \frac{\tilde{e}}{mc} \mathbf{s} \cdot \tilde{\mathbf{F}}(\mathbf{x}) \quad (34)$$

with

$$\begin{aligned} \tilde{\mathbf{F}} = & \left(\frac{g_{\tilde{e}}}{2} - 1 + \frac{1}{\gamma} \right) \tilde{\mathbf{B}} - \left(\frac{g_{\tilde{e}}}{2} - 1 \right) \frac{\gamma}{\gamma + 1} (\boldsymbol{\beta} \cdot \tilde{\mathbf{B}}) \boldsymbol{\beta} \\ & - \left(\frac{g_{\tilde{e}}}{2} - \frac{\gamma}{\gamma + 1} \right) \boldsymbol{\beta} \times \tilde{\mathbf{E}}, \end{aligned} \quad (35)$$

where $\tilde{A} = (\tilde{\phi}, \tilde{\mathbf{A}})$ is the dual 4-vector potential which gives $\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu$ and $\tilde{F}^{\mu\nu} := 1/2 \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual field strength (i.e. $\tilde{\mathbf{B}} = -\mathbf{E}$ and $\tilde{\mathbf{E}} = \mathbf{B}$).

Equation (23) and the TBMT equation given in Eq. (24) are derived as the requirement of covariant is considered. They are classical (non-quantum) equations and we wonder whether the Hamiltonian given in Eq. (32) is consistent with that in the relativistic quantum theory for a charged point particle with intrinsic spin.

The relativistic quantum theory of a spin-1/2 particle is described by the Dirac equation. The Dirac bispinor however has both the particle and antiparticle components, which are entangled by the Dirac equation.

In order to compare with the TBMT equation, we consider the low-energy limit in which the relevant energy is much smaller than the Dirac energy gap E_g and the FW transformation is used to block-diagonalize the Dirac Hamiltonian. In Sec. IV, we will show that the FW transformation of the Dirac Hamiltonian indeed agrees perfectly with the TBMT equation up to the 7th order of $1/E_g$ with the intrinsic spin given by $\mathbf{s} = \hbar \boldsymbol{\sigma}/2$ (σ_i are the Pauli matrices) and the gyromagnetic ratio given by $g_e = 2$. This can be easily generalized for a Dirac dyon by adding the magnetic charge (and we will have $g_e = g_{\tilde{e}} = 2$).

As the Dirac equation always yields $g_e = 2$, we will not see the second term in Eq. (28), which accounts for change of the longitudinal polarization. In order to see that the quantum theory is in accord with the TBMT equation even for the case of $g_e \neq 2$, we study the Dirac-Pauli equation in Sec. V with the inclusion of anomalous dipole moments. The results again affirms the consistency between the FW transformation of the Dirac-Pauli Hamiltonian and the TBMT equation up to the 7th order of $1/E_g$.

IV. FOLDY-WOUTHUYSEN TRANSFORMATION FOR THE DIRAC HAMILTONIAN

The relativistic quantum theory of a Dirac particle is described by the Dirac equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle, \quad (36)$$

where the Dirac bispinor $|\psi\rangle = (\chi, \varphi)^T$ is composed of two 2-component Weyl spinors χ and φ corresponding to

the particle and antiparticle parts. The Dirac Hamiltonian is given by

$$H = mc^2 \tilde{\beta} + c \tilde{\boldsymbol{\alpha}} \cdot \boldsymbol{\Pi} + V, \quad (37)$$

where the 4×4 matrices $\tilde{\beta} = \sigma_z \otimes \mathbf{1}$ and $\tilde{\alpha}_i = \sigma_x \otimes \sigma_i$ are given in the Pauli-Dirac representation [1] and satisfy⁴

$$\begin{aligned} \{\tilde{\beta}, \tilde{\alpha}_i\} &= 0, \\ \{\tilde{\alpha}_i, \tilde{\alpha}_j\} &= 2\delta_{ij}, \\ \tilde{\alpha}_i^2 &= \tilde{\beta}^2 = \mathbf{1}. \end{aligned} \quad (38)$$

The mass of the particle is m and $\boldsymbol{\Pi}$ is the kinetic momentum.

In order to have the proper intrinsic electric dipole moment and the proper intrinsic magnetic dipole moment at the same time, we consider a Dirac dyon (i.e. a Dirac particle with both electric charge e and magnetic charge \tilde{e}). For a dyon [11], the kinetic momentum is given by

$$\boldsymbol{\Pi} = \mathbf{p} - \frac{e}{c} \mathbf{A} - \frac{\tilde{e}}{c} \tilde{\mathbf{A}}, \quad (39)$$

and the scalar potential V is composed of electric and magnetic monopole potentials:

$$V = e\phi + \tilde{e}\tilde{\phi}. \quad (40)$$

In the following, we will first perform the successive FW transformations of the Dirac Hamiltonian up to the 7th order of $1/E_g$ in Sec. IV A, and later show that the results agree with the TBMT equation in Sec. IV B.

A. Foldy-Wouthuysen transformation

In order to perform the FW transformation [2], we have to rewrite the Dirac Hamiltonian to the form:

$$H = E_g \frac{\tilde{\beta}}{2} + \Omega_o + \Omega_E, \quad (41)$$

where $E_g = 2mc^2$ is the Dirac energy gap, and the odd matrix Ω_o and the even matrix Ω_E are defined as

$$\{\tilde{\beta}, \Omega_o\} = 0, \quad [\tilde{\beta}, \Omega_E] = 0. \quad (42)$$

In the case of Eq. (37), we have

$$\Omega_o = c \tilde{\boldsymbol{\alpha}} \cdot \boldsymbol{\Pi}, \quad \Omega_E = V. \quad (43)$$

The resulting effective hamiltonian H_{FW} can be obtained by the successive unitary transformations which partitioning off the odd matrices to a higher order. In

⁴ To avoid confusion with the boost velocity β , we use the checked notation $\tilde{\beta}$ to denote the 4×4 matrix. With the same style, the notations $\tilde{\alpha}$, $\tilde{\gamma}$ [defined in Eq. (88)] and $\tilde{\eta}$ [defined in Eq. (95)] are checked as well.

general, we can use the FW matrix U_{FW} as a single transformation, and expand the exponent of the matrix in powers of $1/E_g$; this is the well-known Löwdin partitioning method [3]. It can be shown that the FW transformation of Eq. (41), namely the transformed Hamiltonian denoted as $H_{\text{FW}} = U_{\text{FW}} H U_{\text{FW}}^{-1}$, up to terms of the 7th order in $1/E_g$ can be written as (see Appendix A)

$$H_{\text{FW}} = \frac{\check{\beta} E_g}{2} + \Omega_E + \sum_{\ell=1}^6 H_{\text{FW}}^{(\ell)} + o(1/E_g^7), \quad (44)$$

where the first four terms $H_{\text{FW}}^{(\ell)}$, $\ell = 1, 2, 3, 4$ are given by

$$H_{\text{FW}}^{(1)} = \frac{\check{\beta} \Omega_o^2}{E_g}, \quad (45a)$$

$$H_{\text{FW}}^{(2)} = \frac{1}{E_g^2} \left(\frac{\mathcal{W}}{2} \right), \quad (45b)$$

$$H_{\text{FW}}^{(3)} = \frac{1}{E_g^3} \left\{ -\check{\beta} \Omega_o^4 + \check{\beta} (\check{\beta} \mathcal{D})^2 \right\}, \quad (45c)$$

$$H_{\text{FW}}^{(4)} = \frac{1}{E_g^4} \left(\frac{1}{24} [[\Omega_o, \mathcal{W}] \Omega_o] - \frac{4}{3} [\mathcal{D}, \Omega_o^3] \right), \quad (45d)$$

and the operators \mathcal{W} and \mathcal{D} are defined as

$$\mathcal{D} = [\Omega_o, \Omega_E], \quad (46a)$$

$$\mathcal{W} = [\mathcal{D}, \Omega_o]. \quad (46b)$$

By using Eq. (43), it can be shown the three Hamiltonians $H_{\text{FW}}^{(\ell=1,2,3)}$ are in agreement with the previous results [2, 14]. The term of the 5th order is given by

$$\begin{aligned} E_g^5 H_{\text{FW}}^{(5)} &= \frac{1}{144} [(\check{\beta} \Omega_o)_{(5)}, \Omega_o] + \frac{1}{2} \sum_{\substack{\ell, m=1 \\ (\ell+m=4)}}^3 [\check{\beta} \mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] \\ &+ \frac{1}{2} \sum_{\ell, m=1}^2 \sum_{n=0}^1 [\check{\beta} \mathcal{O}^{(\ell)}, [\check{\beta} \mathcal{O}^{(m)}, h^{(n)}]], \\ &(\ell+m+n=3) \end{aligned} \quad (47)$$

where the subscript (5) in the commutator $[(\check{\beta} \Omega_o)_{(5)}, \Omega_o]$ indicates that the commutation of $\check{\beta} \Omega_o$ with Ω_o is performed successively by five times; i.e., $[(\check{\beta} \Omega_o)_{(5)}, \Omega_o] = [\check{\beta} \Omega_o, [\check{\beta} \Omega_o, [\check{\beta} \Omega_o, [\check{\beta} \Omega_o, [\check{\beta} \Omega_o, \Omega_o]]]]]$. The term of the 6th order is

$$\begin{aligned} E_g^6 H_{\text{FW}}^{(6)} &= \frac{1}{720} [(\check{\beta} \Omega_o)_{(6)}, \Omega_E] + \frac{1}{2} \sum_{\substack{\ell, m=1 \\ (\ell+m=5)}}^4 [\check{\beta} \mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] \\ &+ \frac{1}{2} \sum_{\ell, m=1}^3 \sum_{n=0}^2 [\check{\beta} \mathcal{O}^{(\ell)}, [\check{\beta} \mathcal{O}^{(m)}, h^{(n)}]], \\ &(\ell+m+n=4) \end{aligned} \quad (48)$$

where the odd matrices $\mathcal{O}^{(\ell)}$ for $\ell = 1, 2, 3, 4$ are given by

$$\begin{aligned} \mathcal{O}^{(1)} &= \check{\beta} \mathcal{D}, \quad \mathcal{O}^{(2)} = -\frac{4}{3} \Omega_o^3, \\ \mathcal{O}^{(3)} &= \frac{1}{6} \check{\beta} [\Omega_o, \mathcal{W}], \quad \mathcal{O}^{(4)} = \frac{8}{15} \Omega_o^5, \end{aligned} \quad (49)$$

and the even matrices $h^{(n)}$ for $n = 0, 1, 2$ are

$$\begin{aligned} h^{(0)} &= \Omega_E, \quad h^{(1)} = \check{\beta} \Omega_o^2, \\ h^{(2)} &= \frac{\mathcal{W}}{2}. \end{aligned} \quad (50)$$

To obtain the FW transformed Hamiltonian $H_{\text{FW}}^{(\ell)}$ up to the 7th order of $1/E_g$, we need only three successive transformations $U_{\text{FW}} = \exp(S_3) \exp(S_2) \exp(S_1)$ (see Appendix A), and it can be shown that S_1 , S_2 and S_3 are all anti-hermitian matrices.

To simplify the calculation, some restrictions and assumptions are made. The electromagnetic field is assumed to be static, as has been used in obtaining Eq. (44). In order to demonstrate the equivalence clearly between the TBMT Hamiltonian and H_{FW} , we further assume that the external fields \mathbf{E} and \mathbf{B} are homogeneous, and thus the field gradient vanishes. Furthermore, the terms proportional to products of field strengths, such as $E_i E_j$, $E_i B_j$ and $B_i B_j$, are all neglected as a good approximation for weak fields.

We now evaluate each term of $H_{\text{FW}}^{(\ell)}$. The kinetic term Ω_o^2/E_g in Eq. (45a) can be written as

$$\begin{aligned} \frac{\Omega_o^2}{E_g} &= \frac{(c\check{\alpha} \cdot \mathbf{\Pi})^2}{2mc^2} \\ &= \frac{1}{2m} \{ |\mathbf{\Pi}|^2 + i\check{\Sigma} \cdot (\mathbf{\Pi} \times \mathbf{\Pi}) \}, \end{aligned} \quad (51)$$

where $[\check{\alpha}_i, \check{\alpha}_j] = 2i\epsilon_{ijk}\Sigma_k$ is used. By using the definition of magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and dual magnetic field $\tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{A}}$, we have $c\mathbf{\Pi} \times \mathbf{\Pi} = i\hbar(e\mathbf{B} + \tilde{e}\tilde{\mathbf{B}})$. By applying the duality $\tilde{\mathbf{B}} = -\mathbf{E}$ in Eq. (51), Eq. (45a) then gives

$$H_{\text{FW}}^{(1)} = \frac{\check{\beta} |\mathbf{\Pi}|^2}{2m} - \check{\beta} \left(\frac{e\hbar}{2mc} \check{\Sigma} \right) \cdot \mathbf{B} - \check{\beta} \left(-\frac{\tilde{e}\hbar}{2mc} \check{\Sigma} \right) \cdot \mathbf{E}. \quad (52)$$

The second term of Eq. (52) is the Zeeman Hamiltonian for an electron (with $e = -|e|$) [15], and the third term is its duality. It is interesting to note that $\tilde{e}\hbar\check{\Sigma}/2mc$ plays the role of electric dipole moment because it couples to the electric field. In this sense, we can define the (proper) intrinsic electric dipole moment ($\mu_p^{\tilde{e}}$) and (proper) intrinsic magnetic dipole moment ($\mu_m^{\tilde{e}}$) as

$$\mu_m^{\tilde{e}} = \frac{e\hbar}{2mc} \check{\Sigma}, \quad (53a)$$

$$\mu_p^{\tilde{e}} = -\frac{\tilde{e}\hbar}{2mc} \check{\Sigma}. \quad (53b)$$

Equation (53a) implies that the dyon's intrinsic gyromagnetic ratio $g_e = 2$ for the Dirac Hamiltonian. We also find

the same gyroelectric ratio $g_e = 2$ for the dyon's intrinsic electric dipole moment.

We now focus on the 2nd-order Hamiltonian $H_{\text{FW}}^{(2)}$ in which the spin-orbit coupled term is included. It can be shown that \mathcal{W} is given by $\mathcal{W} = -2c^2\hbar\mathbf{\Sigma} \cdot ((e\mathbf{E} + \tilde{e}\tilde{\mathbf{E}}) \times \mathbf{\Pi})$ and Eq. (45b) can be written as

$$H_{\text{FW}}^{(2)} = -\frac{1}{2}\mathbf{E} \cdot \left(\frac{\mathbf{\Pi}}{mc} \times \boldsymbol{\mu}_m^{\prime e} \right) - \frac{1}{2}\mathbf{B} \cdot \left(-\frac{\mathbf{\Pi}}{mc} \times \boldsymbol{\mu}_p^{\prime \tilde{e}} \right), \quad (54)$$

where the duality $\tilde{\mathbf{E}} = \mathbf{B}$ is used. The first term of Eq. (54) is the spin-orbit interaction for an electron [15]. On the other hand, we neglect the terms proportional to products of field strengths in evaluating the two terms in Eq. (45c), and thus we can obtain

$$\begin{aligned} H_{\text{FW}}^{(3)} \approx & -\frac{\check{\beta}|\mathbf{\Pi}|^4}{8m^3c^2} + \frac{1}{2}\check{\beta} \left(\frac{|\mathbf{\Pi}|}{mc} \right)^2 (\boldsymbol{\mu}_m^{\prime e} \cdot \mathbf{B}) \\ & + \frac{1}{2}\check{\beta} \left(\frac{|\mathbf{\Pi}|}{mc} \right)^2 (\boldsymbol{\mu}_p^{\prime \tilde{e}} \cdot \mathbf{E}), \end{aligned} \quad (55)$$

where the assumption of homogeneous electromagnetic fields is used, and thus the operator $|\mathbf{\Pi}|^2$ commutes with the magnetic field \mathbf{B} . If the magnetic charge \tilde{e} vanishes, the first term of Eq. (55) is the relativistic mass correction that contributes to the spectrum of fine structure [15].

The second term of Eq. (55) is the relativistic correction to the Zeeman Hamiltonian appearing in $H_{\text{FW}}^{(1)}$ [Eq. (52)]. For the 4th order $H_{\text{FW}}^{(4)}$, it can be shown that

$$\begin{aligned} & \frac{1}{24}[[\Omega_o, \mathcal{W}], \Omega_o] - \frac{4}{3}[[\Omega_o, \Omega_e], \Omega_o^3] \\ & = -\frac{11}{8}(\Omega_o^2\mathcal{W} + \mathcal{W}\Omega_o^2) - \frac{5}{4}\Omega_o\mathcal{W}\Omega_o. \end{aligned} \quad (56)$$

If we neglect all terms proportional to products of electromagnetic fields, one can show that (see Appendix B):

$$(\Omega_o^2\mathcal{W} + \mathcal{W}\Omega_o^2) \approx 2c^2|\mathbf{\Pi}|^2\mathcal{W}, \quad (57a)$$

$$\Omega_o\mathcal{W}\Omega_o \approx -c^2|\mathbf{\Pi}|^2\mathcal{W}. \quad (57b)$$

Note that there is a minus sign in Eq. (57b). The 4th-order term Eq. (45d) with substitution of Eqs. (56) and (57) can be written as

$$H_{\text{FW}}^{(4)} \approx \frac{c^2}{E_g^4} \left(-\frac{3}{2} \right) |\mathbf{\Pi}|^2\mathcal{W} = -\frac{3}{4} \left(\frac{|\mathbf{\Pi}|}{mc} \right)^2 H_{\text{FW}}^{(2)}. \quad (58)$$

It is interesting to note that the 4th order Hamiltonian $H_{\text{FW}}^{(4)}$ is in relation to the 2nd order Hamiltonian $H_{\text{FW}}^{(2)}$

by a relativistic correction $-3(|\mathbf{\Pi}|/mc)^2/4$. For those terms in the 5th order, it can be shown that each term corresponding to Eq. (47) is given by

$$\begin{aligned} & [(\check{\beta}\Omega_o)_{(5)}, \Omega_o] = 32\check{\beta}\Omega_o^6, \\ & \sum_{\substack{\ell, m=1 \\ (\ell+m=4)}}^3 [\check{\beta}\mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] = -\frac{1}{3}\check{\beta}\{\mathcal{D}, [\Omega_o, \mathcal{W}]\} + \frac{32}{9}\check{\beta}\Omega_o^6, \\ & \sum_{\substack{\ell, m=1 \\ (\ell+m+n=3)}}^2 \sum_{n=0}^1 [\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]] \\ & = -\frac{7}{3}\check{\beta}\{\mathcal{D}, [\Omega_o, \mathcal{W}]\} + \frac{18}{3}\{\mathcal{D}, \Omega_o\mathcal{D}\Omega_o\}. \end{aligned} \quad (59)$$

We note that the operator \mathcal{D} is proportional to $\check{\boldsymbol{\alpha}} \cdot \mathbf{E}$, which is of the 1st order of the electric field as well as the operator \mathcal{W} . We find that the terms $\{\mathcal{D}, [\Omega_o, \mathcal{W}]\}$ and $\{\mathcal{D}, \Omega_o\mathcal{D}\Omega_o\}$ in Eq. (59) are proportional to the product of only electric field, and thus will be neglected. On the other hand, the magnetic field in Ω_o^2 is also of the 1st order [see Eq. (51)]. If we further neglect those terms proportional to the products of magnetic field and consider the homogeneous field, Ω_o^6 becomes

$$\begin{aligned} & \frac{\Omega_o^6}{E_g^5} = \frac{(\Omega_o^2)^3}{E_g^5} \\ & \approx \frac{1}{32}mc^2 \left(\frac{|\mathbf{\Pi}|}{mc} \right)^6 - \frac{3}{16} \left(\frac{|\mathbf{\Pi}|}{mc} \right)^4 \cdot (\boldsymbol{\mu}_p^{\prime e} \cdot \mathbf{B} + \boldsymbol{\mu}_m^{\prime \tilde{e}} \cdot \mathbf{E}). \end{aligned} \quad (60)$$

Therefore, $H_{\text{FW}}^{(5)}$ with substitution of Eqs. (59) and (60) becomes

$$\begin{aligned} H_{\text{FW}}^{(5)} \approx & \left(\frac{32}{144} + \frac{32}{18} \right) \frac{\check{\beta}\Omega_o^6}{E_g^5} = \frac{2\check{\beta}\Omega_o^6}{E_g^5} \\ & = \frac{1}{16}mc^2 \left(\frac{|\mathbf{\Pi}|}{mc} \right)^6 - \frac{3}{8} \left(\frac{|\mathbf{\Pi}|}{mc} \right)^4 (\boldsymbol{\mu}_p^{\prime e} \cdot \mathbf{B} + \boldsymbol{\mu}_m^{\prime \tilde{e}} \cdot \mathbf{E}). \end{aligned} \quad (61)$$

The first and second terms of Eq. (61) contribute to the relativistic mass correction and the Zeeman effect, respectively. For the 6th-order term $H_{\text{FW}}^{(6)}$, it can be shown that the commutators of the form $[\check{\beta}\mathcal{O}^\ell, \mathcal{O}^{(m)}]$ are given by

$$\begin{aligned}
[\check{\beta}\mathcal{O}^{(1)}, \mathcal{O}^{(4)}] &= \frac{8}{15}(\Omega_o^4\mathcal{W} + \Omega_o^3\mathcal{W}\Omega_o + \Omega_o^2\mathcal{W}\Omega_o^2 + \Omega_o\mathcal{W}\Omega_o^3 + \mathcal{W}\Omega_o^4), \\
[\check{\beta}\mathcal{O}^{(2)}, \mathcal{O}^{(3)}] &= -\frac{2}{9}(-\Omega_o^4\mathcal{W} + \Omega_o^3\mathcal{W}\Omega_o + \Omega_o\mathcal{W}\Omega_o^3 - \mathcal{W}\Omega_o^4),
\end{aligned} \tag{62}$$

where we also have $[\check{\beta}\mathcal{O}^{(3)}, \mathcal{O}^{(2)}] = [\check{\beta}\mathcal{O}^{(2)}, \mathcal{O}^{(3)}]$ and $[\check{\beta}\mathcal{O}^{(4)}, \mathcal{O}^{(1)}] = [\check{\beta}\mathcal{O}^{(1)}, \mathcal{O}^{(4)}]$. On the other hand, the commutators of the form $[\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]]$ in $H_{\text{FW}}^{(6)}$ are given by

$$\begin{aligned}
[(\check{\beta}\Omega_o)_{(6)}, h^{(0)}] &= \Omega_o^4\mathcal{W} - 4\Omega_o^3\mathcal{W}\Omega_o + 6\Omega_o^2\mathcal{W}\Omega_o^2 - 4\Omega_o\mathcal{W}\Omega_o^3 + \mathcal{W}\Omega_o^4, \\
[\check{\beta}\mathcal{O}^{(1)}, [\check{\beta}\mathcal{O}^{(3)}, h^{(0)}]] &= \frac{1}{6}[\mathcal{D}, [[\Omega_o, \mathcal{W}], \Omega_E]], \\
[\check{\beta}\mathcal{O}^{(2)}, [\check{\beta}\mathcal{O}^{(2)}, h^{(0)}]] &= \frac{16}{9}(\Omega_o^4\mathcal{W} + 2\Omega_o^3\mathcal{W}\Omega_o + 3\Omega_o^2\mathcal{W}\Omega_o^2 + 2\Omega_o\mathcal{W}\Omega_o^3 + \mathcal{W}\Omega_o^4), \\
[\check{\beta}\mathcal{O}^{(1)}, [\check{\beta}\mathcal{O}^{(2)}, h^{(1)}]] &= \frac{8}{3}(\Omega_o^4\mathcal{W} + \Omega_o^3\mathcal{W}\Omega_o + \Omega_o^2\mathcal{W}\Omega_o^2 + \Omega_o\mathcal{W}\Omega_o^3 + \mathcal{W}\Omega_o^4), \\
[\check{\beta}\mathcal{O}^{(2)}, [\check{\beta}\mathcal{O}^{(1)}, h^{(1)}]] &= \frac{4}{3}(\Omega_o^4\mathcal{W} + \Omega_o^3\mathcal{W}\Omega_o + 2\Omega_o^2\mathcal{W}\Omega_o^2 + \Omega_o\mathcal{W}\Omega_o^3 + \mathcal{W}\Omega_o^4), \\
[\check{\beta}\mathcal{O}^{(1)}, [\check{\beta}\mathcal{O}^{(1)}, h^{(2)}]] &= \frac{1}{2}[\mathcal{D}, [\mathcal{D}, \mathcal{W}]].
\end{aligned} \tag{63}$$

Because \mathcal{W} is of the 1st order of an electric field as well as \mathcal{D} , we can use Eq. (57) to reduce these equations into a form with only fields of the 1st order. For example, the term $\Omega_o^3\mathcal{W}\Omega_o$ becomes $\Omega_o^3\mathcal{W}\Omega_o = \Omega_o^2(\Omega_o\mathcal{W}\Omega_o) \approx c^4|\mathbf{\Pi}|^2(-|\mathbf{\Pi}|^2\mathcal{W})$. On the other hand, $[\mathcal{D}, [\mathcal{D}, \mathcal{W}]]$ and $[\mathcal{D}, [[\Omega_o, \mathcal{W}], \Omega_E]]$ are neglected because they are at least of the 2nd order of fields. In that sense, by the use of Eqs. (62) and (63), one can obtain

$$\begin{aligned}
[(\check{\beta}\Omega_o)_{(6)}, \Omega_E] &\approx 16c^4|\mathbf{\Pi}|^4\mathcal{W}, \\
\sum_{\substack{\ell, m=1 \\ (\ell+m=5)}}^4 [\check{\beta}\mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] &\approx \frac{128}{45}c^4|\mathbf{\Pi}|^4\mathcal{W}, \\
\sum_{\substack{\ell, m=1 \\ (\ell+m=n=4)}}^3 \sum_{n=0}^2 [\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]] &\approx \frac{64}{9}c^4|\mathbf{\Pi}|^4\mathcal{W}.
\end{aligned} \tag{64}$$

Therefore, Eq. (48) with substitution of Eq. (64) becomes

$$\begin{aligned}
H_{\text{FW}}^{(6)} &\approx \frac{1}{E_g^6} \left[\frac{16}{720} + \frac{1}{2} \left(\frac{128}{45} \right) + \frac{1}{2} \left(\frac{64}{9} \right) \right] |\mathbf{\Pi}|^4\mathcal{W} \\
&= \frac{5c^4}{E_g^6} |\mathbf{\Pi}|^4\mathcal{W} \\
&= \frac{5}{8} \left(\frac{|\mathbf{\Pi}|}{mc} \right)^4 \left(\frac{\mathcal{W}}{2E_g^2} \right).
\end{aligned} \tag{65}$$

Eq. (65) is the relativistic correction to the spin-orbit in-

teraction in $H_{\text{FW}}^{(2)}$. Therefore, $H_{\text{FW}}^{(1)}$ and $H_{\text{FW}}^{(3)}$ and $H_{\text{FW}}^{(5)}$ are composed of kinetic energy, interaction energy of Zeeman effect and their relativistic corrections. On the other hand, $H_{\text{FW}}^{(2)}$ and $H_{\text{FW}}^{(4)}$ and $H_{\text{FW}}^{(6)}$ contain only spin-orbit interaction and its relativistic corrections.

To simplify the expression of $H_{\text{FW}}^{(\ell)}$, we can define a scaled kinetic momentum operator ξ as⁵

$$\xi = \frac{\mathbf{\Pi}}{mc}. \tag{66}$$

By replacing $|\mathbf{\Pi}|/mc$ with Eq. (66), Eq. (44) with substitution of Eqs. (52), (54), (55), (58), (61) and (65) becomes a sum of two terms:

$$H_{\text{FW}} \approx H_{\text{orbit}} + H_{\text{spin}}, \tag{67}$$

where H_{orbit} is the kinetic energy plus the potential energy, namely

$$H_{\text{orbit}} = \check{\beta}mc^2 \left(1 + \frac{1}{2}|\xi|^2 - \frac{1}{8}|\xi|^4 + \frac{1}{16}|\xi|^6 \right) + V, \tag{68}$$

and H_{spin} is the energy of intrinsic dipole moments placing in electromagnetic fields, namely,

⁵ It must be stressed that the operator ξ does not directly correspond to the Lorentz boost velocity β given in the previous section. The appropriate transformation between ξ and opera-

tor for the boost velocity is considered in Sec.IV B.

$$H_{\text{spin}} = -\mathbf{E} \cdot \left[\check{\beta} \boldsymbol{\mu}_p^{\prime \tilde{e}} + \frac{1}{2} (\boldsymbol{\xi} \times \boldsymbol{\mu}_m^{\prime e}) \right] - \mathbf{B} \cdot \left[\check{\beta} \boldsymbol{\mu}_m^{\prime e} - \frac{1}{2} (\boldsymbol{\xi} \times \boldsymbol{\mu}_p^{\prime \tilde{e}}) \right] \\ + \check{\beta} \left(\frac{1}{2} |\boldsymbol{\xi}|^2 - \frac{3}{8} |\boldsymbol{\xi}|^4 \right) (\boldsymbol{\mu}_m^{\prime e} \cdot \mathbf{B} + \boldsymbol{\mu}_p^{\prime \tilde{e}} \cdot \mathbf{E}) + \left(-\frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) \left\{ -\frac{1}{2} \mathbf{E} \cdot (\boldsymbol{\xi} \times \boldsymbol{\mu}_m^{\prime e}) - \frac{1}{2} \mathbf{B} \cdot (-\boldsymbol{\xi} \times \boldsymbol{\mu}_p^{\prime \tilde{e}}) \right\}. \quad (69)$$

In Sec. IV B, we will focus on the dipole Hamiltonian [Eq. (69)]. We will show that Eq. (69) is in agreement with TBMT equation, provided that the proper transformation of ξ and Lorentz boost velocity β is taken care of.

B. In relation to TBMT equation

We will show that the FW transformation of the Dirac Hamiltonian of a dyon is equivalent to the Hamiltonian obtained from TBMT equation with $g_e = g_{\tilde{e}} = 2$. That is, Eq. (68) is equivalent to Eq. (33) and Eq. (69) to Eq. (34) with Eq. (28) and Eq. (35) for $g_e = g_{\tilde{e}} = 2$. However, we must first find the boost velocity in order to compare them. It must be emphasized that β in TBMT equation is the boost velocity but ξ in H_{FW} is not. One has to define the boost operator $\hat{\beta}$ via

$$\xi = \frac{\hat{\beta}}{\sqrt{1 - |\hat{\beta}|^2}}, \quad \hat{\gamma} \equiv \frac{1}{\sqrt{1 - |\hat{\beta}|^2}}, \quad (70)$$

because the kinetic momentum $\Pi \equiv mc\xi = m\mathbf{U}$ and the 4-velocity $U^\alpha = (\gamma c, \gamma\beta)$. By using Eq. (70), the kinetic energy operator Eq. (68) behaves like $mc^2 (1 + \frac{1}{2} |\boldsymbol{\xi}|^2 - \frac{1}{8} |\boldsymbol{\xi}|^4 + \frac{1}{16} |\boldsymbol{\xi}|^4) = mc^2 (1 + \frac{1}{2} |\hat{\beta}|^2 + \frac{3}{8} |\hat{\beta}|^4 + \frac{5}{16} |\hat{\beta}|^6 + o(8))$. On the other hand, the expansion of Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$ with respect to small boost β is $\gamma = 1 + \frac{1}{2} \beta^2 + \frac{3}{8} \beta^4 + \frac{5}{16} \beta^6 + o(8)$. This implies that the kinetic energy operator corresponds to the classical relativistic energy γmc^2 , as expected. The boost operator $\hat{\beta}$ plays an important role on showing the equivalence between H_{spin} and TBMT Hamiltonian. For an electron, Eq. (69) without a magnetic charge ($\tilde{e} = 0$) becomes

$$H_{\text{spin}}^{(\tilde{e}=0)} = -\mathbf{E} \cdot \left[\frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) (\boldsymbol{\xi} \times \boldsymbol{\mu}_m^{\prime e}) \right] - \mathbf{B} \cdot \left[\check{\beta} \left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) \boldsymbol{\mu}_m^{\prime e} \right] \\ = -\boldsymbol{\mu}_m^{\prime e} \cdot \left[\check{\beta} \left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) \mathbf{B} - \frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) \boldsymbol{\xi} \times \mathbf{E} \right]. \quad (71)$$

The first term in the right hand side of the first equality of Eq. (71) is an effective electric dipole moment caused by the boosted intrinsic spin magnetic moment, which is the spin-orbit interaction. The second one is the Zeeman term. Nevertheless, Eq. (71) provides the relativistic correction to the Zeeman and spin-orbit interactions. For the Zeeman term, the non-relativistic limit up to $1/mc$ is

$$H_{\text{Zeeman}} = -\check{\beta} \boldsymbol{\mu}_m^{\prime e} \cdot \mathbf{B}, \quad (72)$$

which is the same as the interaction of a classical magnetic moment and a magnetic field. To the 4th order of

ξ , the relativistic correction to H_{Zeeman} is

$$H_{\text{Zeeman}} = -\check{\beta} \left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) \boldsymbol{\mu}_m^{\prime e} \cdot \mathbf{B}. \quad (73)$$

On the other hand, the spin-orbit interaction denoted as H_{so} is

$$H_{\text{so}} = \frac{1}{2} \boldsymbol{\mu}_m^{\prime e} \cdot \boldsymbol{\xi} \times \mathbf{E} \\ = \frac{|e|\hbar}{4m^2 c^2} \boldsymbol{\Sigma} \cdot \mathbf{E} \times \boldsymbol{\Pi}, \quad (74)$$

where $e = -|e|$ is used in the second equality, and the relativistic correction to this term is

$$H_{\text{so}} = \frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) \boldsymbol{\mu}_m^{\prime e} \cdot \boldsymbol{\xi} \times \mathbf{E}. \quad (75)$$

We now go back to the discussion of $H_{\text{spin}}^{\tilde{e}=0}$ and TBMT equation. In order to compare Eq. (71) with TBMT Hamiltonian, we have to transform $\boldsymbol{\xi}$ in Eq. (71) to $\hat{\boldsymbol{\beta}}$. Using Eq. (70), we have

$$\left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) = 1 - \frac{|\hat{\boldsymbol{\beta}}|^2}{2} - \frac{|\hat{\boldsymbol{\beta}}|^4}{8} + o(6), \quad (76a)$$

$$\frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) |\boldsymbol{\xi}| = \frac{|\hat{\boldsymbol{\beta}}|}{2} - \frac{|\hat{\boldsymbol{\beta}}|^3}{8} - \frac{|\hat{\boldsymbol{\beta}}|^5}{16} + o(7). \quad (76b)$$

The effective spin magnetic moment in TBMT Hamiltonian [Eq. (30)] transforms like $(1/\gamma) \boldsymbol{\mu}_m^{\prime e}$, and we have

$$\frac{1}{\gamma} = 1 - \frac{\beta^2}{2} - \frac{\beta^4}{8} + o(6), \quad (77)$$

which is exactly the same as Eq. (76a) up to terms of the 4th order in β . On the other hand, the effective electric dipole moment transforms like $(g_e/2 - \gamma/(\gamma+1))$, and g_e factor in the Dirac Hamiltonian is always 2. We obtain

$$\left(1 - \frac{\gamma}{1+\gamma} \right) \beta = \frac{\beta}{2} - \frac{\beta^3}{8} - \frac{\beta^5}{16} + o(7), \quad (78)$$

which is exactly the same as Eq. (76b) up to terms of the 5th order in β . Because g_e factor equals 2, the longitudinal term $\boldsymbol{\mu}_m^{\prime e} \cdot \boldsymbol{\xi}$ disappears in both TBMT Hamiltonian and $H_{\text{spin}}^{(\tilde{e}=0)}$. In the following, we will use the following two approximations directly:

$$\begin{aligned} \left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) &\approx \frac{1}{\gamma}, \\ \frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) |\boldsymbol{\xi}| &\approx \left(1 - \frac{\hat{\gamma}}{\hat{\gamma}+1} \right). \end{aligned} \quad (79)$$

Therefore, we show that up to the fifth order of boost velocity β , the Dirac Hamiltonian of an electron is equivalent to the TBMT Hamiltonian which is obtained from the requirement of covariance form of classical spin. This implies that in the FW representation, after summing over all infinite expansion terms, the Dirac Hamiltonian of an electron would be of the form

$$H_{\text{spin}}^{\tilde{e}=0} = -\boldsymbol{\mu}_m^{\prime e} \cdot \left[\check{\beta} \frac{1}{\hat{\gamma}} \mathbf{B} - \left(1 - \frac{\hat{\gamma}}{1+\hat{\gamma}} \right) \hat{\boldsymbol{\beta}} \times \mathbf{E} \right] \quad (80)$$

for the spin part, and of the form

$$H_{\text{orbit}} = \hat{\gamma} \check{\beta} m c^2 + V \quad (81)$$

for orbital part. The effective magnetic field in Eq. (80) is the same as Eq. (30) with Eq. (28) for $g_e = 2$.

Furthermore, for the FW transformation of the Dirac Hamiltonian [Eq. (67)], the TBMT equation can be generalized to include an effective spin magnetic moment resulting from the boosted intrinsic electric dipole moment. To the 1st order in $|\boldsymbol{\xi}| = |\mathbf{\Pi}|/mc$, we find that the effective dipole moments transform like

$$\begin{aligned} (\boldsymbol{\mu}_p^{\tilde{e}})_{\text{eff}} &\approx \check{\beta} \boldsymbol{\mu}_p^{\tilde{e}} + \frac{1}{2} (\boldsymbol{\xi} \times \boldsymbol{\mu}_m^{\prime e}), \\ (\boldsymbol{\mu}_m^e)_{\text{eff}} &\approx \check{\beta} \boldsymbol{\mu}_m^e - \frac{1}{2} (\boldsymbol{\xi} \times \boldsymbol{\mu}_p^{\tilde{e}}). \end{aligned} \quad (82)$$

This means that an intrinsic electric dipole moment can result in an effective magnetic dipole moment when it is moving. Nevertheless, a moving spin magnetic moment can also intrinsically induce an effective electric dipole moment. Consider higher orders of the boost velocity, we rewrite Eq. (69) as

$$\begin{aligned} H_{\text{spin}} = -\mathbf{E} \cdot &\left[\check{\beta} \left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) \boldsymbol{\mu}_p^{\tilde{e}} + \frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) (\boldsymbol{\xi} \times \boldsymbol{\mu}_m^{\prime e}) \right] \\ &- \mathbf{B} \cdot \left[\check{\beta} \left(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4 \right) \boldsymbol{\mu}_m^e - \frac{1}{2} \left(1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4 \right) (\boldsymbol{\xi} \times \boldsymbol{\mu}_p^{\tilde{e}}) \right]. \end{aligned} \quad (83)$$

It is shown that intrinsic dipole moments transform like $(1 - \frac{1}{2} |\boldsymbol{\xi}|^2 + \frac{3}{8} |\boldsymbol{\xi}|^4) \approx 1/\hat{\gamma}$ and the boosted dipole moments transform as $\frac{1}{2} (1 - \frac{3}{4} |\boldsymbol{\xi}|^2 + \frac{5}{8} |\boldsymbol{\xi}|^4) \approx (1 - \frac{\hat{\gamma}}{\hat{\gamma}+1})$ (see Eq. (79)). Therefore, $(\boldsymbol{\mu}_p^{\tilde{e}})_{\text{eff}}$ and $(\boldsymbol{\mu}_m^e)_{\text{eff}}$ do not form a second rank tensor in the sense that their transformation is not a covariant form like Eq. (13), but the following

form:

$$\begin{aligned} (\boldsymbol{\mu}_p^{\tilde{e}})_{\text{eff}} &\approx \check{\beta} \frac{1}{\hat{\gamma}} \boldsymbol{\mu}_p^{\tilde{e}} + \left(1 - \frac{\hat{\gamma}}{\hat{\gamma}+1} \right) (\hat{\boldsymbol{\beta}} \times \boldsymbol{\mu}_m^{\prime e}), \\ (\boldsymbol{\mu}_m^e)_{\text{eff}} &\approx \check{\beta} \frac{1}{\hat{\gamma}} \boldsymbol{\mu}_m^e - \left(1 - \frac{\hat{\gamma}}{\hat{\gamma}+1} \right) (\hat{\boldsymbol{\beta}} \times \boldsymbol{\mu}_p^{\tilde{e}}). \end{aligned} \quad (84)$$

This implies that an energy caused by dipole moments in the description of Dirac Hamiltonian is not simply the contraction of tensorial dipole density and field tensor: $H_{\text{spin}} \neq -\boldsymbol{\mu}_p \cdot \mathbf{E} - \boldsymbol{\mu}_m \cdot \mathbf{B}$, in which $\boldsymbol{\mu}_p$ and $\boldsymbol{\mu}_m$ transform as in Eq. (13). As a result, H_{spin} is not a Lorentz scalar.

In short, in this section we have shown that up to terms of the 7th order in $1/E_g$, the FW transformation of the Dirac Hamiltonian of an electron is in agreement with TBMT Hamiltonian [Eq. (30)] with $g_e = 2$. The result can be generalized to a particle with an intrinsic electric dipole moment. Because of the duality of electromagnetic fields, a Dirac dyon would manifest this feature. Furthermore, we also find the relativistic corrections to the Zeeman term and spin-orbit interaction

V. FOLDY-WOUTHUYSEN TRANSFORMATION FOR THE DIRAC-PAULI HAMILTONIAN

In Sec. IV, we have shown that, up to the 7th order in $1/E_g$, the FW transformation of the Dirac Hamiltonian for a dyon is in agreement with Eq. (26) and Eq. (27) for $g_e = g_{\bar{e}} = 2$. Since the Dirac Hamiltonian automatically yields $g_e = 2$ and $g_{\bar{e}} = 2$, the second term in Eq. (28) and Eq. (35) vanishes and thus the longitudinal polarization does not change. In order to see that the relativistic quantum theory of a spin-1/2 particle is in accord with the TBMT equation even when the change rate of the longitudinal polarization is concerned, we have to study the spin-1/2 particle with anomalous magnetic dipole moment (AMM) and anomalous electric dipole moment (AEM).

The relativistic quantum theory of a spin-/2 dyon with the inclusion of AMM and AEM can be described by the Dirac-Pauli equation [9, 12]

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \mathcal{H} |\psi\rangle, \quad (85)$$

where the Dirac-Pauli Hamiltonian \mathcal{H} is the Dirac Hamiltonian H [given in Eq. (37)] augmented with the corrections for the AMM and AEM:

$$\mathcal{H} = H + \mu'(-\check{\beta}\boldsymbol{\Sigma} \cdot \mathbf{B} + i\check{\gamma} \cdot \mathbf{E}) + d'(\check{\beta}\boldsymbol{\Sigma} \cdot \mathbf{E} + i\check{\gamma} \cdot \mathbf{B}). \quad (86)$$

The coefficients μ' and d' are defined as follows

$$\mu' = \left(\frac{g_e}{2} - 1\right) \frac{e\hbar}{2mc}, \quad d' = \left(\frac{g_{\bar{e}}}{2} - 1\right) \frac{\tilde{e}\hbar}{2mc}, \quad (87)$$

which measures the AMM and AEM, respectively (note $\mu' = 0$ for $g_e = 2$ and $d' = 0$ for $g_{\bar{e}} = 2$). The 4×4 matrices $\check{\beta}\boldsymbol{\Sigma}$ and $\check{\gamma}$ are defined as

$$\check{\beta}\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \check{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad (88)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices. We will see that the Dirac-Pauli Hamiltonian given in Eq. (86) is

compatible to generic values of g_e and $g_{\bar{e}}$ and thus can accommodate AMM and AEM.

In order to obtain the FW transformation of Eq. (86), we have to rewrite Eq. (86) in terms of odd and even matrices. According to Eq. (42), we have

$$\mathcal{H} = \check{\beta}mc^2 + \Omega_E^A + \Omega_o^A, \quad (89)$$

where the superscript A indicates the inclusion of anomalous dipole moments. We note that μ' and d' are of the 1st order of $1/E_g$. Therefore, Ω_E^A and Ω_o^A can be written as

$$\begin{aligned} \Omega_E^A &= \Omega_E + \frac{\Omega_E^f}{E_g}, \\ \Omega_o^A &= \Omega_o + \frac{\Omega_o^f}{E_g}, \end{aligned} \quad (90)$$

where Ω_E and Ω_o are given in Eq. (43) and

$$\begin{aligned} \Omega_E^f &= \check{\beta}\boldsymbol{\Sigma} \cdot (-\mu''\mathbf{B} + d''\mathbf{E}), \\ \Omega_o^f &= i\check{\gamma} \cdot (\mu''\mathbf{E} + d''\mathbf{B}), \\ \mu'' &= E_g\mu', \quad d'' = E_gd'. \end{aligned} \quad (91)$$

The superscript f indicates that these terms are of the 1st order of electromagnetic fields. Because we consider only those terms proportional to the 1st order of fields, the products of fields will be neglected. The validity of Eq. (A17) is still true provided that the odd term of the second FW transformation denoted as \mathcal{O}' in Eq. (89) starts from $1/E_g^3$. This can be seen as follows. After the first FW transformation, $S_1 = \check{\beta}\Omega_o^A/E_g$, Eq. (89) becomes

$$\mathcal{H}' = \frac{\check{\beta}}{2}E_g + h + \mathcal{O}, \quad (92)$$

where h and \mathcal{O} are even and odd terms, respectively. It can be shown that the odd term \mathcal{O} can be written as

$$\mathcal{O} = \frac{\mathcal{O}^{(1)}}{E_g} + \frac{\mathcal{O}^{(2)}}{E_g^2} + \frac{\mathcal{O}^{(3)}}{E_g^3} + \frac{\mathcal{O}^{(4)}}{E_g^4} + o\left(\frac{1}{E_g^5}\right), \quad (93)$$

where the corresponding $\mathcal{O}^{(n)}$, $n = 1, 2, 3, 4$, are given by

$$\begin{aligned} \mathcal{O}^{(1)} &= \check{\beta}[\Omega_o, \Omega_E], \\ \mathcal{O}^{(2)} &= 2c\check{\beta}\check{\eta}(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \boldsymbol{\Pi} - \frac{4}{3}\Omega_o^3, \\ \mathcal{O}^{(3)} &= -\frac{4}{3}(\Omega_o\{\Omega_o, \Omega_o^f\} + \Omega_o^f\Omega_o^2) + \frac{1}{6}\check{\beta}[\Omega_o, \mathcal{W}], \\ \mathcal{O}^{(4)} &= -\frac{4}{3}c^3\check{\beta}\check{\eta}(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \boldsymbol{\Pi}|\boldsymbol{\Pi}|^2 + \frac{8}{15}\Omega_o^5, \end{aligned} \quad (94)$$

where the matrix $\check{\eta}$ is defined as

$$\check{\eta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (95)$$

If g_e and $g_{\bar{e}}$ are both equal to 2, then Eq. (94) goes back to Eq. (49). Using the second FW transformation $S_2 =$

$\check{\beta}\mathcal{O}/E_g$ on Eq. (93), we can obtain the other odd term denoted as \mathcal{O}' . The 1st order term $\mathcal{O}'^{(1)}$ is zero because \mathcal{O} starts from the 1st order of $1/E_g$ at least. $\mathcal{O}'^{(2)}$ can be written as $\mathcal{O}'^{(2)} = [\check{\beta}\mathcal{O}^{(1)}, h^{(0)}]$. Nevertheless, we have $\mathcal{O}^{(1)} = \check{\beta}[\Omega_o, \Omega_E] = \check{\beta}i\hbar\check{\alpha} \cdot (e\mathbf{E} + \bar{e}\mathbf{B})$ and $h^{(0)} = \Omega_E = V$ is a scalar that commutes with $\mathcal{O}^{(1)}$, and thus $\mathcal{O}'^{(2)}$ vanishes. This implies that Eq. (A17) is still valid in this case. The matrix h in Eq. (93) is given by

$$\begin{aligned} h = & \Omega_E^A + \frac{1}{2E_g}[\check{\beta}\Omega_o^A, \Omega_o^A] + \frac{1}{2E_g^2}[(\check{\beta}\Omega_o^A)_{(2)}, \Omega_E^A] \\ & + \frac{1}{8E_g^3}[(\check{\beta}\Omega_o^A)_{(3)}, \Omega_o^A] + \frac{1}{24}[(\check{\beta}\Omega_o^A)_{(4)}, \Omega_E^A] \\ & + \frac{1}{144E_g^5}[(\check{\beta}\Omega_o^A)_{(5)}, \Omega_o^A] + \frac{1}{720E_g^6}[(\check{\beta}\Omega_o^A)_{(6)}, \Omega_E^A]. \end{aligned} \quad (96)$$

To obtain h' , we need extra corrections to h , as shown in Eq. (A18). The resulting Hamiltonian can be written as

$$\mathcal{H}_{\text{FW}} = H_{\text{FW}} + \mathcal{H}_{\text{FW1}} + \mathcal{H}_{\text{FW2}}. \quad (97)$$

The Hamiltonian H_{FW} is given in Eq. (67), \mathcal{H}_{FW1} contains those terms proportional to $(-\mu''\mathbf{B} + d''\mathbf{E})$, and \mathcal{H}_{FW2} contains those terms proportional to $(\mu''\mathbf{E} + d''\mathbf{B})$. Focusing on the term proportional to $(-\mu'\mathbf{B} + d'\mathbf{E})$, namely, $[(\check{\beta}\Omega_o^A)_{(n=2,4)}, \Omega_E^A]$ in Eq. (96). For $n = 2$, we have

$$\begin{aligned} [(\check{\beta}\Omega_o^A)_{(2)}, \Omega_E^A] &= [\check{\beta}\Omega_o^A, [\check{\beta}\Omega_o^A, \Omega_E^A]] \\ &= [[\Omega_o^A, \Omega_E^A], \Omega_o^A] \\ &= \mathcal{W} + \frac{\mathcal{W}^f}{E_g}, \end{aligned} \quad (98)$$

where \mathcal{W} is given in Eq. (46) and \mathcal{W}^f is defined as

$$\begin{aligned} \mathcal{W}^f &= [[\Omega_o, \Omega_E^f], \Omega_o] \\ &= -4c^2\check{\beta}\Sigma \cdot \Pi(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \Pi, \end{aligned} \quad (99)$$

which is proportional to electric and magnetic fields. By using Eq. (99), the term $[(\check{\beta}\Omega_o^A)_{(4)}, \Omega_E^A]$ can be written as

$$\begin{aligned} &[(\check{\beta}\Omega_o^A)_{(4)}, \Omega_E^A] \\ &= [\check{\beta}\Omega_o^A, [\check{\beta}\Omega_o^A, \mathcal{W} + \frac{\mathcal{W}^f}{E_g}]] \\ &= [[\Omega_o, \mathcal{W}], \Omega_o] - \frac{4c^2}{E_g}\mathcal{W}^f|\Pi|^2. \end{aligned} \quad (100)$$

The first term of Eq. (100) is just one of the 4th order terms of $H_{\text{FW}}^{(4)}$ shown in Sec. II. It is important to note

that the second term of Eq. (100) is collected in $h^{(5)}$, not $h^{(4)}$. It can be shown that the only term that contributes to $\Sigma \cdot \Pi(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \Pi|\Pi|^2$ is $[\check{\beta}\mathcal{O}^{(2)}, \mathcal{O}^{(2)}]$ that is the correction term in $h^{(5)}$ (see Eq. (A18)). Using $\mathcal{O}^{(2)}$ in Eq. (94), we have

$$\begin{aligned} &\frac{1}{2}[\check{\beta}\mathcal{O}^{(2)}, \mathcal{O}^{(2)}] \\ &= \check{\beta}(\mathcal{O}^{(2)})^2 \\ &= \check{\beta}\left(2c\check{\beta}\check{\eta}(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \Pi - \frac{4}{3}\Omega_o^3\right)^2 \\ &= \check{\beta}\frac{16}{9}\Omega_o^6 - \frac{8}{3}c^4(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \Pi|\Pi|^2[\check{\eta}, \check{\alpha}_\ell]\Pi_\ell. \end{aligned} \quad (101)$$

It can be shown that $[\check{\eta}, \check{\alpha}_\ell] = -2\check{\beta}\Sigma_\ell$, and thus we obtain

$$\begin{aligned} &\frac{1}{2}[\check{\beta}\mathcal{O}^{(2)}, \mathcal{O}^{(2)}] \\ &= \check{\beta}\frac{16}{9}\Omega_o^6 + \frac{16}{3}c^4\check{\beta}\Sigma \cdot \Pi(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \Pi|\Pi|^2. \end{aligned} \quad (102)$$

Consider those terms proportional to $(-\mu''\mathbf{B} + d''\mathbf{E})$: one comes from Eq. (99) in the corresponding term and the other is obtained from Ω_E^f , Eq. (100) and Eq. (102). Using the definitions of $\xi = \Pi/mc$ (see Eq. (70)), $\mu'' = E_g\mu'$ and $d'' = E_gd'$, one obtain

$$\begin{aligned} \mathcal{H}_{\text{FW1}} &= \frac{1}{2E_g^3}\mathcal{W}^f + \check{\beta}\Sigma \cdot (-\mu'\mathbf{B} + d'\mathbf{E}) \\ &\quad + \frac{c^4}{E_g^5}\left(\frac{16}{3} + \frac{16}{24}\right)\check{\beta}\Sigma \cdot \Pi(-\mu''\mathbf{B} + d''\mathbf{E}) \cdot \Pi|\Pi|^2 \\ &= \left(-\frac{1}{2} + \frac{3}{8}|\xi|^2\right)\check{\beta}\Sigma \cdot \xi(-\mu'\mathbf{B} + d'\mathbf{E}) \cdot \xi \\ &\quad + \check{\beta}\Sigma \cdot (-\mu'\mathbf{B} + d'\mathbf{E}). \end{aligned} \quad (103)$$

For $g_e = 2$ and $g_{\bar{e}} = 2$, \mathcal{H}_{FW1} vanishes. We now transform ξ in Eq. (103) in terms of the boost velocity $\hat{\beta}$. By using the transformation between ξ and $\hat{\beta}$ [see Eq. (70)], it can be shown that $(\frac{1}{2} - \frac{3}{8}|\xi|^2)|\xi|^2 \approx \frac{\hat{\gamma}}{1+\hat{\gamma}}|\hat{\beta}|^2$. Substituting AMM coefficient $\mu' = (\frac{g_e}{2} - 1)\frac{e\hbar}{2mc}$ and AEM coefficient $d' = (\frac{g_{\bar{e}}}{2} - 1)\frac{e\hbar}{2mc}$ into Eq. (103), we find that \mathcal{H}_{FW1} can be written as

$$\begin{aligned}
\mathcal{H}_{\text{FW1}} &= -\frac{\hat{\gamma}}{\hat{\gamma}+1} \check{\beta} \Sigma \cdot \hat{\beta} (-\mu' \mathbf{B} + d' \mathbf{E}) \cdot \hat{\beta} + \check{\beta} \Sigma \cdot (-\mu' \mathbf{B} + d' \mathbf{E}) \\
&= -\frac{\hat{\gamma}}{\hat{\gamma}+1} \check{\beta} \left[-\left(\frac{g_e}{2} - 1\right) \mu_m'^e \cdot \hat{\beta} \mathbf{B} \cdot \hat{\beta} - \left(\frac{g_{\bar{e}}}{2} - 1\right) \mu_p'^{\bar{e}} \cdot \hat{\beta} \mathbf{E} \cdot \hat{\beta} \right] + \check{\beta} \left[-\left(\frac{g_e}{2} - 1\right) \mu_m'^e \cdot \mathbf{B} - \left(\frac{g_{\bar{e}}}{2} - 1\right) \mu_p'^{\bar{e}} \cdot \mathbf{E} \right].
\end{aligned} \tag{104}$$

On the other hand, the terms proportional to $(\mu'' \mathbf{E} + d'' \mathbf{B})$ correspond to $\{\Omega_o, \Omega_o^f\} = 2c\beta \Sigma \cdot \Pi \times (\mu'' \mathbf{E} + d'' \mathbf{B})$. Similar to the derivation for \mathcal{H}_{FW1} , we collect all terms proportional to $\{\Omega_o, \Omega_o^f\}$ and obtain

$$\mathcal{H}_{\text{FW2}} = \check{\beta} \{\Omega_o, \Omega_o^f\} \frac{1}{E_g^2} \left(1 - \frac{1}{2} |\xi|^2 + \frac{3}{8} |\xi|^4 \right). \tag{105}$$

Using $\mu'' = E_g \mu'$ and $d'' = E_g d'$, we have

$$\mathcal{H}_{\text{FW2}} = \left(1 - \frac{1}{2} |\xi|^2 + \frac{3}{8} |\xi|^4 \right) \Sigma \cdot \xi \times (\mu' \mathbf{E} + d' \mathbf{B}). \tag{106}$$

By using Eq. (70), we find that

$$\begin{aligned}
\mathcal{H}_{\text{FW2}} &= \Sigma \cdot \hat{\beta} \times (\mu' \mathbf{E} + d' \mathbf{B}) \\
&= \left(\frac{g_e}{2} - 1\right) \mu_m'^e \cdot (\hat{\beta} \times \mathbf{E}) - \left(\frac{g_{\bar{e}}}{2} - 1\right) \mu_p'^{\bar{e}} \cdot (\hat{\beta} \times \mathbf{B}).
\end{aligned} \tag{107}$$

To focus on the interaction of dipole moments and external fields, we have to combine H_{spin} , \mathcal{H}_{FW1} and \mathcal{H}_{FW2} together. After a straightforward calculations, we find that

$$\begin{aligned}
H_{\text{spin}} + \mathcal{H}_{\text{FW1}} + \mathcal{H}_{\text{FW2}} &= -\mu_m'^e \cdot \left\{ \left(\frac{g_e}{2} - 1 + \frac{1}{\hat{\gamma}}\right) \check{\beta} \mathbf{B} - \left(\frac{g_e}{2} - \frac{\hat{\gamma}}{\hat{\gamma}+1}\right) \hat{\beta} \times \mathbf{E} - \frac{\hat{\gamma}}{\hat{\gamma}+1} \left(\frac{g_e}{2} - 1\right) \hat{\beta} (\mathbf{B} \cdot \hat{\beta}) \right\} \\
&\quad - \mu_p'^{\bar{e}} \cdot \left\{ \left(\frac{g_{\bar{e}}}{2} - 1 + \frac{1}{\hat{\gamma}}\right) \check{\beta} \mathbf{E} + \left(\frac{g_{\bar{e}}}{2} - \frac{\hat{\gamma}}{\hat{\gamma}+1}\right) \hat{\beta} \times \mathbf{B} - \frac{\hat{\gamma}}{\hat{\gamma}+1} \left(\frac{g_{\bar{e}}}{2} - 1\right) \hat{\beta} (\mathbf{E} \cdot \hat{\beta}) \right\}.
\end{aligned} \tag{108}$$

Eq. (108) is in agreement with Eq. (34) when the replacement $\gamma \rightarrow \hat{\gamma}$ and the duality transformation for electromagnetic fields are used. Without magnetic charge, Eq. (108) coincides with Eq. (28) for arbitrary values of g_e . The dual part of the TBMT equation for spin is also obtained.

In short, the whole derivations in this section have assumed that electromagnetic fields are static and homogeneous. Therefore, we show that up to terms of the 7th order in $1/E_g$, the FW transformation including anomalous dipole moments coincides with the TBMT equation for the spin-1/2 particle with arbitrary g_e and $g_{\bar{e}}$.

VI. CONCLUSIONS AND DISCUSSION

To investigate the low-energy limit of the relativistic quantum theory for a spin-1/2 charged particle, which is described by the Dirac equation, we perform a series of successive FW transformations on the Dirac Hamiltonian up to terms of the 7th order in $1/E_g$. Assuming the electromagnetic fields are static and homogeneous, and taking care of the relation between the kinematic momentum Π used in the Dirac Hamiltonian and the boost

velocity β used in the TBMT equation, we show that the resulting FW transformation of the Dirac Hamiltonian is in agreement with the classical orbital Hamiltonian H_{orbit} plus the TBMT Hamiltonian H_{spin} with the gyromagnetic ratio g_e being equal to 2. Through electromagnetic duality, this can be generalized for a spin-1/2 dyon, which has both electric and magnetic charges and thus possesses both intrinsic magnetic dipole moment $\mu_m'^e$ and intrinsic electric dipole moment $\mu_p'^{\bar{e}}$.

To affirm the consistency between the low-energy limit of the relativistic quantum theory and the classical counterpart to a broader extent, we consider the relativistic quantum theory for a spin-1/2 dyon with arbitrary values of the gyromagnetic and gyroelectric ratios, which is described by the Dirac-Pauli equation, namely, the Dirac equation with augmentation for AMM and AEM. Up the 7th order in $1/E_g$ again, we show that the FW transformation of the Dirac-Pauli Hamiltonian is also in accord with $H_{\text{orbit}} + H_{\text{spin}}$.

Many phenomena regarding spin dynamics have been observed and can be explained by the TBMT equation. These include the anomalous Zeeman effect, spin-orbit interaction, Thomas precession and change rate of the longitudinal polarization (see Sec. 11.11 of [13] for a brief

review). The TBMT equation is however derived merely by the requirement of covariance without invoking any first principles. By studying the FW transformation of the Dirac Hamiltonian and the Dirac-Pauli Hamiltonian, we have shown that the TBMT equation as a phenomenological formula is in fact supported by the first principle of the fundamental relativistic quantum theory as a low-energy limit. (The relativistic quantum theory further requires the spin to be quantized as $\mathbf{s} = \hbar \boldsymbol{\sigma}/2$; this result cannot be obtained at the phenomenological level.) Therefore, the correspondence principle is again shown to be established.

By far, the agreement between the Dirac/Dirac-Pauli equation and the orbital equation plus the TBMT equation is only proven up to the 7th order in $1/E_g$. Further research is needed to investigate the FW transformation to higher orders and a generic expression for the FW transformation at an arbitrary order could be obtained by mathematical induction. If this is the case, performing successive FW transformations ad infinitum is expected to yield the result in precise agreement with the orbital equation plus the TBMT equation.

Furthermore, the assumption of static and homogeneous fields can be released. In time-varying and/or inhomogeneous fields, the TBMT equation has to be generalized to allow gradient force terms like $(\boldsymbol{\mu}_m \cdot \nabla) \mathbf{B}$ and the FW transformation of the Dirac/Dirac-Pauli Hamiltonian shall yield the corresponding terms accordingly. The gradient force terms should not be missing, as $(\boldsymbol{\mu}_m \cdot \nabla) \mathbf{B}$ is used in the Stern-Gerlach experiment to separate spin-up and spin-down particles. Furthermore, the detailed investigation for the dipole moments interacting with the time-variation of electromagnetic fields may predict new physics. However, the calculation for the FW transformation will be much more complicated if the fields are non-static and inhomogeneous.

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Appendix A: Foldy-Wouthuysen transformation

In this appendix, we expand the Dirac Hamiltonian up to terms of the 7th order in $1/E_g$ with $E_g = 2mc^2$. The Dirac Hamiltonian can be separated into two parts. One is the even operator denoted as Ω_E , which commutes with $\tilde{\beta}$, and the other is odd operator Ω_o , which anti-commutes with $\tilde{\beta}$:

$$\begin{aligned} [\tilde{\beta}, \Omega_E] &= 0, \\ \{\tilde{\beta}, \Omega_o\} &= 0. \end{aligned} \quad (\text{A1})$$

The Dirac Hamiltonian can be written as

$$H = \frac{\tilde{\beta}}{2} E_g + \Omega_E + \Omega_o, \quad (\text{A2})$$

where $\Omega_E = e\phi + \tilde{e}\tilde{\phi}$ and $\Omega_o = c\tilde{\boldsymbol{\alpha}} \cdot \boldsymbol{\Pi}$. The kinetic momentum $\boldsymbol{\Pi}$ is $\boldsymbol{\Pi} = \mathbf{p} - \frac{e}{c} \mathbf{A} - \frac{\tilde{e}}{c} \tilde{\mathbf{A}}$. The first transformation operator can be written as

$$U_1 = e^{S_1}, \quad S_1 = \tilde{\beta} \Omega_o / E_g. \quad (\text{A3})$$

The Dirac Hamiltonian under the unitary transformation U_1 can be written as

$$H_1 = U_1 H U_1^{-1} = \frac{\tilde{\beta}}{2} E_g + \Omega_E + \sum_{n=1}^{\infty} \frac{1}{E_g^n} \left\{ \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\tilde{\beta} \Omega_o)_{(n)}, \Omega_o] + \frac{1}{n!} [(\tilde{\beta} \Omega_o)_{(n)}, \Omega_E] \right\}, \quad (\text{A4})$$

where the subscript n at $(\tilde{\beta} \Omega_o)_{(n)}$ is defined as, for example, $[(\tilde{\beta} \Omega_o)_{(3)}, \Omega_o] = [\tilde{\beta} \Omega_o, [\tilde{\beta} \Omega_o, [\tilde{\beta} \Omega_o, \Omega_E]]]$. Equation (A4) can be again separated into odd and even parts. The even part of Eq. (A4) denoted as h can be written

as

$$h = h^{(0)} + \sum_{n=1}^{\infty} \frac{h^{(n)}}{E_g^n}, \quad (\text{A5})$$

where $h^{(0)} = \Omega_E$ and

$$\begin{aligned} h^{(n=1,3,5,\dots)} &= \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\Omega_o)_{(n)}, \Omega_o], \\ h^{(n=2,4,6,\dots)} &= \frac{1}{n!} [(\check{\beta}\Omega_o)_{(n)}, \Omega_E]. \end{aligned} \quad (\text{A6})$$

The odd part of Eq. (A4) denoted as \mathcal{O} can be written as

$$\mathcal{O} = \sum_{n=1}^{\infty} \frac{\mathcal{O}^{(n)}}{E_g^n}, \quad (\text{A7})$$

where

$$\begin{aligned} \mathcal{O}^{(n=1,3,5,\dots)} &= \frac{1}{n!} [(\check{\beta}\Omega_o)_{(n)}, \Omega_E], \\ \mathcal{O}^{(n=2,4,6,\dots)} &= \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\Omega_o)_{(n)}, \Omega_o]. \end{aligned} \quad (\text{A8})$$

Therefore, Eq. (A4) becomes

$$H_1 = \frac{\check{\beta}}{2} E_g + h + \mathcal{O}, \quad (\text{A9})$$

where h contains those terms with only even matrices and \mathcal{O} contains only odd matrices. The second transformation denoted as $U_2 = \exp(S_2)$, where S_2 is $S_2 = \check{\beta}\mathcal{O}/E_g$. We have

$$\begin{aligned} H_2 &= U_2 H_1 U_2^{-1} = \frac{\beta}{2} E_g + h + \sum_{n=1}^{\infty} \frac{1}{E_g^n} \left\{ \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\mathcal{O})_{(n)}, \mathcal{O}] + \frac{1}{n!} [(\check{\beta}\mathcal{O})_{(n)}, h] \right\} \\ &= \frac{\beta}{2} E_g + h' + \mathcal{O}', \end{aligned} \quad (\text{A10})$$

where h' and \mathcal{O}' are the new even and odd parts, respectively, of the right hand side of the first equality. The even part of Eq. (A10) denoted as h' can be written as

$$\begin{aligned} h' &= h + \left\{ \begin{aligned} &\sum_{n=1,3,5,\dots} \frac{1}{E_g^n} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\mathcal{O})_{(n)}, \mathcal{O}] \\ &\sum_{n=2,4,6,\dots} \frac{1}{E_g^n} \frac{1}{n!} [(\check{\beta}\mathcal{O})_{(n)}, h] \end{aligned} \right\} \\ &= h^{(0)} + \sum_{m=1}^{\infty} \frac{h'^{(m)}}{E_g^m}. \end{aligned} \quad (\text{A11})$$

The odd term \mathcal{O}' in Eq. (A10) is given by

$$\begin{aligned} \mathcal{O}' &= \left\{ \begin{aligned} &\sum_{n=2,4,6,\dots} \frac{1}{E_g^n} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\mathcal{O})_{(n)}, \mathcal{O}] \\ &\sum_{n=1,3,5,\dots} \frac{1}{E_g^n} \frac{1}{n!} [(\check{\beta}\mathcal{O})_{(n)}, h] \end{aligned} \right\} \\ &= \sum_{m=3}^{\infty} \frac{\mathcal{O}'^{(m)}}{E_g^m}. \end{aligned} \quad (\text{A12})$$

Firstly, we note that $\mathcal{O}'^{(1)}$ and $\mathcal{O}'^{(2)}$ in Eq. (A12) vanish as we have m starting from 3. For the former result, the reason is that the lowest order of the unprimed odd term \mathcal{O} is 1, and thus in the second line of the first equality in Eq. (A12), the primed odd term \mathcal{O}' is at least of the 2nd order. On the other hand, the explicit form of $\mathcal{O}'^{(2)}$ can be written as $\mathcal{O}'^{(2)} = [\check{\beta}\mathcal{O}^{(1)}, h^{(0)}]$. However, $\mathcal{O}^{(1)}$ is given by Eq. (A8) for $n = 1$, namely, $\mathcal{O}^{(1)} = \check{\beta}[\Omega_o, \Omega_E] = i\hbar\check{\alpha} \cdot (e\mathbf{E} + \tilde{e}\tilde{\mathbf{E}})$, which commutes with $h^{(0)} = \Omega_E = e\phi + \tilde{e}\tilde{\phi}$. If we perform the third transformation which is $S_3 = \check{\beta}\mathcal{O}'/E_g$ at $H_2 = \frac{\check{\beta}E_g}{2} + h' + \mathcal{O}'$, we will obtain a new even term h'' as well as the new odd term \mathcal{O}'' : $H_3 = U_3 H_2 U_3^{-1} = \frac{\check{\beta}}{2} E_g + h'' + \mathcal{O}''$.

The even term h'' is given by

$$\begin{aligned}
 h'' &= h' + \left\{ \begin{aligned} &\sum_{n=1,3,5,\dots} \frac{1}{E_g^n} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\mathcal{O}')_{(n)}, \mathcal{O}'] \\ &\sum_{n=2,4,6,\dots} \frac{1}{E_g^n} \frac{1}{n!} [(\check{\beta}\mathcal{O}')_{(n)}, h'] \end{aligned} \right. \\
 &= h''^{(0)} + \sum_{m=1}^{\infty} \frac{h''^{(m)}}{E_g^m}.
 \end{aligned} \tag{A13}$$

The odd term \mathcal{O}'' is given by

$$\begin{aligned}
 \mathcal{O}'' &= \left\{ \begin{aligned} &\sum_{n=2,4,6,\dots} \frac{1}{E_g^n} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) [(\check{\beta}\mathcal{O}')_{(n)}, \mathcal{O}'] \\ &\sum_{n=1,3,5,\dots} \frac{1}{E_g^n} \frac{1}{n!} [(\check{\beta}\mathcal{O}')_{(n)}, h'] \end{aligned} \right. \\
 &= \sum_{m=4}^{\infty} \frac{\mathcal{O}''^{(m)}}{E_g^m}.
 \end{aligned} \tag{A14}$$

The odd term \mathcal{O}'' starts from $m = 4$. Obviously, $\mathcal{O}''^{(1)}$ is zero because $\mathcal{O}'^{(m)}$ starts from $m = 3$. This can also be seen as follows. The explicit form of the next three terms of $\mathcal{O}''^{(m)}$ are

$$\begin{aligned}
 \mathcal{O}''^{(2)} &= [\check{\beta}\mathcal{O}'^{(1)}, h'^{(0)}], \\
 \mathcal{O}''^{(3)} &= [\check{\beta}\mathcal{O}'^{(2)}, h'^{(0)}] + [\check{\beta}\mathcal{O}'^{(1)}, h'^{(1)}], \\
 \mathcal{O}''^{(4)} &= [\check{\beta}\mathcal{O}'^{(1)}, h'^{(2)}] + [\check{\beta}\mathcal{O}'^{(2)}, h'^{(1)}] + [\check{\beta}\mathcal{O}'^{(3)}, h'^{(0)}].
 \end{aligned} \tag{A15}$$

Because $\mathcal{O}'^{(1)}$ and $\mathcal{O}'^{(2)}$ are zero, $\mathcal{O}''^{(2)}$ and $\mathcal{O}''^{(3)}$ vanish. It can be shown that $\mathcal{O}''^{(4)}$ does not vanish. Using Eq. (A13), $h''^{(m)}$ for $m = 1, 2, \dots, 6$ are given by

$$\begin{aligned}
 h''^{(0)} &= h'^{(0)}, \\
 h''^{(1)} &= h'^{(1)}, \\
 h''^{(2)} &= h'^{(2)}, \\
 h''^{(3)} &= h'^{(3)} + \left(1 - \frac{1}{2!}\right) [\check{\beta}\mathcal{O}'^{(1)}, \mathcal{O}'^{(1)}], \\
 h''^{(4)} &= h'^{(4)} + \left(1 - \frac{1}{2!}\right) \left([\check{\beta}\mathcal{O}'^{(1)}, \mathcal{O}'^{(2)}] + [\check{\beta}\mathcal{O}'^{(2)}, \mathcal{O}'^{(1)}] \right) + \frac{1}{2!} [\check{\beta}\mathcal{O}'^{(1)}, [\check{\beta}\mathcal{O}'^{(1)}, h'^{(0)}]], \\
 h''^{(5)} &= h'^{(5)} + \left(1 - \frac{1}{2!}\right) \sum_{\substack{\ell, m=1 \\ (\ell+m=4)}}^3 [\check{\beta}\mathcal{O}'^{(\ell)}, \mathcal{O}'^{(m)}] + \frac{1}{2!} \sum_{\substack{\ell, m=1 \\ (\ell+m+n=3)}}^2 \sum_{n=0}^1 [\check{\beta}\mathcal{O}'^{(\ell)}, [\check{\beta}\mathcal{O}'^{(m)}, h'^{(n)}]], \\
 h''^{(6)} &= h'^{(6)} + \left(1 - \frac{1}{2!}\right) \sum_{\substack{\ell, m=1 \\ (\ell+m=5)}}^4 [\check{\beta}\mathcal{O}'^{(\ell)}, \mathcal{O}'^{(m)}] + \frac{1}{2!} \sum_{\substack{\ell, m=1 \\ (\ell+m+n=4)}}^3 \sum_{n=0}^2 [\check{\beta}\mathcal{O}'^{(\ell)}, [\check{\beta}\mathcal{O}'^{(m)}, h'^{(n)}]].
 \end{aligned} \tag{A16}$$

Because $\mathcal{O}'^{(1)}$ and $\mathcal{O}'^{(2)}$ are zero, we have $h''^{(3)} = h'^{(3)}$ and $h''^{(4)} = h'^{(4)}$, since the constraints $\ell + m = 4$ and $\ell + m = 5$ imply $(\ell, m) = \{(1, 3), (2, 2), (3, 1)\}$ and $(\ell, m) = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, respectively. The commutator $[\check{\beta}\mathcal{O}'^{(\ell)}, \mathcal{O}'^{(m)}]$ in $h''^{(5)}$ and $h''^{(6)}$ vanishes. Consider the term $[\check{\beta}\mathcal{O}'^{(\ell)}, [\check{\beta}\mathcal{O}'^{(m)}, h'^{(n)}]]$ in $h''^{(5)}$ and $h''^{(6)}$ subject to the constraints $\ell + m + n = 3$ and $\ell + m + n = 4$, respectively. For $n = 0$, we have $(\ell, m) = \{(1, 2), (2, 1)\}$ and $(\ell, m) = \{(1, 3), (2, 2), (3, 1)\}$ and thus $[\check{\beta}\mathcal{O}'^{(\ell)}, [\check{\beta}\mathcal{O}'^{(m)}, h'^{(n=0)}]]$ vanishes in $h''^{(5)}$ and $h''^{(6)}$. For $n = 1$ and $n = 2$, the term $[\check{\beta}\mathcal{O}'^{(\ell)}, [\check{\beta}\mathcal{O}'^{(m)}, h'^{(n)}]]$ still vanishes.

Therefore, up to terms of the 7th order in $\frac{1}{E_g}$, we obtain an important result

$$h''^{(n)} = h'^{(n)}, \quad n = 1, 2, \dots, 6, \quad (\text{A17})$$

and $h'^{(n)}$ (i.e., Eq. (A11)) is given by

$$\begin{aligned} h'^{(0)} &= h^{(0)}, \\ h'^{(1)} &= h^{(1)}, \\ h'^{(2)} &= h^{(2)}, \\ h'^{(3)} &= h^{(3)} + \left(1 - \frac{1}{2!}\right) [\check{\beta}\mathcal{O}^{(1)}, \mathcal{O}^{(1)}], \\ h'^{(4)} &= h^{(4)} + \left(1 - \frac{1}{2!}\right) ([\check{\beta}\mathcal{O}^{(1)}, \mathcal{O}^{(2)}] + [\check{\beta}\mathcal{O}^{(2)}, \mathcal{O}^{(1)}]) + \frac{1}{2!} [\check{\beta}\mathcal{O}^{(1)}, [\check{\beta}\mathcal{O}^{(1)}, h^{(0)}]], \\ h'^{(5)} &= h^{(5)} + \left(1 - \frac{1}{2!}\right) \sum_{\substack{\ell, m=1 \\ (\ell+m=4)}}^3 [\check{\beta}\mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] + \frac{1}{2!} \sum_{\substack{\ell, m=1 \\ (\ell+m+n=3)}}^2 \sum_{n=0}^1 [\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]], \\ h'^{(6)} &= h^{(6)} + \left(1 - \frac{1}{2!}\right) \sum_{\substack{\ell, m=1 \\ (\ell+m=5)}}^4 [\check{\beta}\mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] + \frac{1}{2!} \sum_{\substack{\ell, m=1 \\ (\ell+m+n=4)}}^3 \sum_{n=0}^2 [\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]]. \end{aligned} \quad (\text{A18})$$

Therefore, in order to obtain the FW transformation up to $1/E_g^7$, we need to know four odd terms $\mathcal{O}^{(n)}$ for $n = 1, 2, 3, 4$. On the other hand, if we perform the transformation again by using $S_4 = \check{\beta}\mathcal{O}''/E_g$, since $\mathcal{O}''^{(m)}$ starts from $m = 4$ (i.e. $\mathcal{O}''^{(m)} = 0$ for $m = 1, 2, 3$), the transformation S_4 does not change Eq. (A17). The resulting $\mathcal{O}'''^{(m)}$ will start from at least $m = 5$. To bring the odd term to the 7th order, we need $S_5 = \check{\beta}\mathcal{O}'''/E_g$ and $S_6 = \check{\beta}\mathcal{O}''''/E_g$. However, the two transformations also do not change the validity of Eq. (A17). Therefore the resulting Hamiltonian can be written as

$$\begin{aligned} H_{\text{FW}} &= U_{\text{FW}} H U_{\text{FW}}^{-1} \\ &= \frac{\check{\beta}}{2} E_g + \sum_{n=0}^6 H_{\text{FW}}^{(n)} + o(1/E_g^7). \end{aligned} \quad (\text{A19})$$

By using Eqs. (A6), (A18) and (A17), after straightforward calculations it can be shown that

$$H_{\text{FW}}^{(1)} = \frac{\check{\beta}\Omega_o^2}{E_g}, \quad (\text{A20a})$$

$$H_{\text{FW}}^{(2)} = \frac{1}{E_g^2} \left(\frac{\mathcal{W}}{2} \right), \quad (\text{A20b})$$

$$H_{\text{FW}}^{(3)} = \frac{1}{E_g^3} \left\{ -\check{\beta}\Omega_o^4 + \check{\beta}(\check{\beta}\mathcal{D})^2 \right\}, \quad (\text{A20c})$$

$$H_{\text{FW}}^{(4)} = \frac{1}{E_g^4} \left(\frac{1}{24} [[\Omega_o, \mathcal{W}], \Omega_o] - \frac{4}{3} [\mathcal{D}, \Omega_o^3] \right), \quad (\text{A20d})$$

$$H_{\text{FW}}^{(5)} = \frac{1}{E_g^5} \left\{ \frac{1}{144} [(\check{\beta}\Omega_o)_{(5)}, \Omega_o] + \frac{1}{2} \sum_{\substack{\ell, m=1 \\ (\ell+m=4)}}^3 [\check{\beta}\mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] + \frac{1}{2} \sum_{\substack{\ell, m=1 \\ (\ell+m+n=3)}}^2 \sum_{n=0}^1 [\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]] \right\}, \quad (\text{A20e})$$

$$H_{\text{FW}}^{(6)} = \frac{1}{E_g^6} \left\{ \frac{1}{720} [(\check{\beta}\Omega_o)_{(6)}, \Omega_o] + \frac{1}{2} \sum_{\substack{\ell, m=1 \\ (\ell+m=5)}}^4 [\check{\beta}\mathcal{O}^{(\ell)}, \mathcal{O}^{(m)}] + \frac{1}{2} \sum_{\substack{\ell, m=1 \\ (\ell+m+n=4)}}^3 \sum_{n=0}^2 [\check{\beta}\mathcal{O}^{(\ell)}, [\check{\beta}\mathcal{O}^{(m)}, h^{(n)}]] \right\}, \quad (\text{A20f})$$

$$(\text{A20g})$$

where $\mathcal{D} \equiv [\Omega_o, \Omega_E]$ and $\mathcal{W} \equiv [\mathcal{D}, \Omega_o]$.

Appendix B: Derivation of Equation (57b)

The explicit form of $\Omega_o \mathcal{W} \Omega_o$ [Eq. (57)] plays an important role in obtaining the correct coefficient of each term in $H_{\text{FW}}^{(6)}$. By definition, $\Omega_o = c \tilde{\alpha} \cdot \mathbf{\Pi}$ and $\mathcal{W} = [[\Omega_o, \Omega_E], \Omega_o] = [[c \tilde{\alpha} \cdot \mathbf{\Pi}, V], c \tilde{\alpha} \cdot \mathbf{\Pi}]$, where $V = e\phi + \tilde{e}\tilde{\phi}$. In the derivation for Eq. (57), electromagnetic fields are assumed to be homogeneous and static. This means that the terms involving gradient of fields are neglected. Therefore, we have

$$\frac{1}{c^2} \Omega_o \mathcal{W} \Omega_o = -2c^2 \hbar \epsilon_{pqr} \mathcal{E}_q \Pi_i \Pi_r \Pi_j (\tilde{\alpha}_i \Sigma_p \tilde{\alpha}_j), \quad (\text{B1})$$

where $\mathcal{E}_q \equiv (eE_q + \tilde{e}\tilde{E}_q)$ and

$$\mathcal{W} = -2c^2 \hbar \mathbf{\Sigma} \cdot (\mathcal{E} \times \mathbf{\Pi}) \quad (\text{B2})$$

is used. Furthermore, it can be shown that

$$[\Pi_i, \Pi_j] = \frac{i\hbar}{c} \epsilon_{ijk} \mathcal{B}_k, \quad (\text{B3})$$

were $\mathcal{B}_k = eB_k + \tilde{e}\tilde{B}_k$. By using $[\tilde{\alpha}_i, \sigma_p] = 2i\epsilon_{ipm} \tilde{\alpha}_m$ and $\epsilon_{pqr} \epsilon_{ipm} = \delta_{qm} \delta_{ri} - \delta_{qi} \delta_{rm}$, Eq. (B1) can be written as

$$\begin{aligned} \frac{1}{c^2} \Omega_o \mathcal{W} \Omega_o &= -2\hbar \mathcal{E}_q (2i\tilde{\alpha}_q \tilde{\alpha}_j |\mathbf{\Pi}|^2 \Pi_j - 2i\tilde{\alpha}_r \tilde{\alpha}_j \Pi_q \Pi_r \Pi_j \\ &\quad + \epsilon_{pqr} \Sigma_p \Pi_i \Pi_r \Pi_i + i\epsilon_{pqr} \epsilon_{ijm} \Sigma_m \Sigma_p \Pi_i \Pi_r \Pi_j). \end{aligned} \quad (\text{B4})$$

By using $\tilde{\alpha}_q \tilde{\alpha}_j = \delta_{qj} + i\epsilon_{qj\ell} \Sigma_\ell$, the first term of Eq. (B4) becomes

$$2i\tilde{\alpha}_q \tilde{\alpha}_j |\mathbf{\Pi}|^2 \Pi_j = 2i|\mathbf{\Pi}|^2 \Pi_q - 2\epsilon_{qj\ell} \Sigma_\ell |\mathbf{\Pi}|^2 \Pi_j. \quad (\text{B5})$$

On the other hand, the second term of Eq. (B4) can be written as

$$-2i\tilde{\alpha}_r \tilde{\alpha}_j \Pi_q \Pi_r \Pi_j = -2i\Pi_q |\mathbf{\Pi}|^2 + \frac{2i\hbar}{c} \Sigma_\ell \Pi_q \mathcal{B}_\ell, \quad (\text{B6})$$

where Eq. (B3) is used. The third term of Eq. (B4) can be written as

$$\begin{aligned} &\epsilon_{pqr} \Sigma_p \Pi_i \Pi_r \Pi_i \\ &= \epsilon_{pqr} \Sigma_p \Pi_i \left(\frac{i\hbar}{c} \epsilon_{ril} \mathcal{B}_\ell + \Pi_i \Pi_r \right) \\ &= \frac{i\hbar}{c} (\Sigma_i \Pi_i \mathcal{B}_q - \Sigma_\ell \Pi_q \mathcal{B}_\ell) + \epsilon_{pqr} \Sigma_p |\mathbf{\Pi}|^2 \Pi_r. \end{aligned} \quad (\text{B7})$$

The fourth term of Eq. (B4) becomes

$$\begin{aligned} &i\epsilon_{pqr} \epsilon_{ijm} \Sigma_m \Sigma_p \Pi_i \Pi_r \Pi_j \\ &= i\epsilon_{pqr} \epsilon_{ijm} \Sigma_m \Sigma_p \Pi_i \left(\frac{i\hbar}{c} \epsilon_{rjl} \mathcal{B}_\ell + \Pi_j \Pi_r \right) \\ &= -\frac{\hbar}{c} \epsilon_{pqr} (2\mathcal{B}_\ell \Sigma_\ell \Pi_r - \mathcal{B}_i \Pi_i \Sigma_r + \mathcal{B}_m \Sigma_m) \Sigma_p, \end{aligned} \quad (\text{B8})$$

where $\epsilon_{ijm} \Pi_i \Pi_j = \frac{i\hbar}{c} \mathcal{B}_m$ and $\epsilon_{ijm} \epsilon_{rjl} = \delta_{ir} \delta_{mj} - \delta_{il} \delta_{mr}$ are used. We note that there is a field \mathcal{E}_q in the Eq. (B4). By neglecting the product of fields $\mathcal{E}_i \mathcal{B}_j$, Eq. (B4) with substitutions of Eqs. (B5), (B6), (B7) and (B8) becomes

$$\begin{aligned} \frac{1}{c^2} \Omega_o \mathcal{W} \Omega_o &\approx -2c^2 \hbar \mathcal{E}_q (-2\epsilon_{qj\ell} \Sigma_\ell |\mathbf{\Pi}|^2 \Pi_j + \epsilon_{pqr} \Sigma_p |\mathbf{\Pi}|^2 \Pi_r) \\ &= -|\mathbf{\Pi}|^2 (-2c^2 \epsilon_{pqr} \Sigma_p \mathcal{E}_q \Pi_r) \\ &= -|\mathbf{\Pi}|^2 \mathcal{W}, \end{aligned} \quad (\text{B9})$$

and we have

$$\Omega_o \mathcal{W} \Omega_o \approx -c^2 |\mathbf{\Pi}|^2 \mathcal{W}. \quad (\text{B10})$$

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